Celestial Mechanics – Solutions

Alexander V. Krivov & Tobias Stein[1]

Unit 13

Problem 13.1

Using the formula for the Legendre polynomials,

$$\frac{1}{\sqrt{1-2bx+b^2}} = \sum_{k=0}^{\infty} b^k P_k(x),$$

in the case of an *external* perturber we derived in the lecture

$$\begin{aligned}\frac{1}{\Delta} &= \frac{1}{\sqrt{r'^2+r^2-2rr'\cos H}}\\ &= \left(r'\sqrt{1-2\frac{r}{r'}\cos H+\left(\frac{r}{r'}\right)^2}\right)^{-1}\\ &= \frac{1}{r'}\sum_{k=0}^{\infty}\left(\frac{r}{r'}\right)^k P_k(\cos H),\end{aligned}$$

and found the disturbing function to be

$$\begin{aligned}R &= GM'\left[\frac{1}{\Delta}-\frac{r\cos H}{r'^2}\right]\\ &= GM'\left[\frac{1}{r'}\sum_{k=0}^{\infty}\left(\frac{r}{r'}\right)^k P_k(\cos H)-\frac{r\cos H}{r'^2}\right]\\ &= \frac{GM'}{r'}\sum_{k=2}^{\infty}\left(\frac{r}{r'}\right)^k P_k(\cos H).\end{aligned} \tag{1}$$

(The $k=0$ term disappeared because it is independent of r and the $k=1$ term cancels with the indirect term of R.)

For an *internal* perturber, we shall use expansion in powers of r'/r rather than r/r':

$$\begin{aligned}\frac{1}{\Delta} &= \frac{1}{\sqrt{r'^2+r^2-2rr'\cos H}}\\ &= \frac{1}{r\sqrt{1-2\frac{r'}{r}\cos H+\left(\frac{r'}{r}\right)^2}}\\ &= \frac{1}{r}\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos H),\end{aligned}$$

[1] tobias.stein@uni-jena.de

and the disturbing function for the internal perturber reads

$$\begin{aligned}
R &= GM'\left[\frac{1}{\Delta} - \frac{r\cos H}{r'^2}\right] \\
&= GM'\left[\frac{1}{r}\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos H) - \frac{r\cos H}{r'^2}\right] \\
&= GM'\left[\frac{1}{r} + \frac{r'}{r^2}\cos H + \frac{1}{r}\sum_{k=2}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos H) - \frac{r\cos H}{r'^2}\right] \\
&= GM'\left[\frac{1}{r} + \left(\frac{r'}{r^2} - \frac{r}{r'^2}\right)\cos H + \frac{1}{r}\sum_{k=2}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos H)\right].
\end{aligned} \tag{2}$$

Expansion (2) is similar to (1), except that it contains two additional terms.

Bonus problem 13.2

Neglecting the variations of coefficients j, one can give a lower limit for the number of of terms just by calculating the number of variations for the exponents of e, e', i, and i'. Since the only conditions are $k_1 + k_2 + k_3 + k_4 \leq n$ and $k_1, k_2, k_3, k_4 \geq 0$, this number of terms is given by

$$N = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{i=0}^{n-k-j}\sum_{h=0}^{n-k-j-i} 1. \tag{3}$$

This can be solved step by step, using (e.g., BRONSTEIN et al.: Taschenbuch der Mathematik):

$$\begin{aligned}
\sum_{i=0}^{m} 1 &= m+1, \\
\sum_{i=0}^{m} i &= \frac{m(m+1)}{2}, \\
\sum_{i=0}^{m} i^2 &= \frac{m(m+1)(2m+1)}{6}, \\
\sum_{i=0}^{m} i^3 &= \frac{m^2(m+1)^2}{4}.
\end{aligned}$$

We find

$$N = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} (n-k-j-i+1). \tag{4}$$

$$N = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \left[(n-k-j+1)^2 - \frac{(n-k-j)(n-k-j+1)}{2} \right]$$

$$N = \frac{1}{2} \sum_{k=0}^{n} \sum_{j=0}^{n-k} [(n-k-j+1)(n-k-j+2)] \tag{5}$$

$$N = \frac{1}{2} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \left[j^2 - (2n-2k+3)j + (n-k+1)(n-k+2) \right]$$

$$N = \frac{1}{2} \sum_{k=0}^{n} \left[\frac{(n-k)(n-k+1)(2n-2k+1)}{6} - \frac{(n-k)(n-k+1)(2n-2k+3)}{2} \right.$$
$$\left. + (n-k+1)^2(n-k+2) \right]$$

$$N = \frac{1}{2 \cdot 3} \sum_{k=0}^{n} \left[-2(n-k)(n-k+1)(n-k+2) + 3(n-k+1)^2(n-k+2) \right]$$

$$N = \frac{1}{2 \cdot 3} \sum_{k=0}^{n} [(n-k+1)(n-k+2)(n-k+3)] . \tag{6}$$

Comparing equations (3), (4), (5), and (6), we can guess the recursion and derive

$$N = \frac{1}{2 \cdot 3 \cdot 4} [(n+1)(n+2)(n+3)(n+4)] = \frac{(n+4)!}{4!n!} = \binom{n+4}{4}.$$

So, for $n = 7$, the result is $N = 330$. The actual number of 7^{th}-order terms that Le Verrier found was 469. The discrepancy is due to the fact that not only the exponents k can be varied but also the frequencies j. The case known from the lecture – expansion in e to the order of e^2 – shows that the above estimate is not always a good estimate: one would obtain $N = 15$ for $n = 2$, which is wrong by more than a factor of 2.

Problem 13.3

As covered in the lecture, evolution if complex eccentricity (to 2nd order e and I) is given by

$$z = \underbrace{e_{\text{f}} \exp[i\varpi]}_{\text{const}} + e_{\text{p}} \exp[i(At + \beta)], \tag{7}$$

where the oscillatory term has amplitude e_{p} and phase shift β. The angular frequency is

$$A = \frac{n}{4} \frac{M'}{M_\odot} \alpha^2 b_{3/2}^{1}(\alpha) \approx \frac{3\pi}{2P} \frac{M'}{M_\odot} \alpha^3, \tag{8}$$

where $n = 2\pi/P$ is Earth's Kepler frequency, $P = 1$ yr the orbital period, M' the perturber's mass, $M_\odot$ the Solar mass, $\alpha = a/a'$ the ratio of semi-major axes of Earth vs perturber, and $b(\alpha) \approx 3\alpha$ a Laplace coefficient. For $\alpha_{\text{Jup}} = 1/5.2$ and $M'_{\text{Jup}}/M_\odot = 9.5 \times 10^{-4}$, we find

$$A_{\text{Jup}} = 32 \times 10^{-6} \text{ rad/yr} = 1.8 \times 10^{-3} \,^\circ/\text{yr} = \frac{360^\circ}{0.2 \text{ Myr}}. \tag{9}$$

Jupiter induces secular precession with a full period of 0.2 Myr. For Saturn's $\alpha_{\rm Sat} = 1/9.6$ and $M'_{\rm Sat}/M_\odot = 2.9 \times 10^{-4}$, we find

$$A_{\rm Sat} = 1.5 \times 10^{-6} \text{ rad/yr} = 8.9 \times 10^{-5} \,{}^\circ/\text{yr} = \frac{360^\circ}{4 \text{ Myr}}, \tag{10}$$

corresponding to a period of 4 Myr.