Celestial Mechanics – Solutions

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Unit 12

Problem 12.1

The specific force exerted on planet i by the star (s) and the other $n-1$ planets is given by

$$\ddot{\vec{\rho}}_i = -G\frac{M_\mathrm{s}}{r_i^3}\vec{r}_i - G\sum_{\substack{k=1\\k\neq i}}^{n}\frac{M_k}{|\vec{r}_i-\vec{r}_k|^3}(\vec{r}_i-\vec{r}_k),$$

that on the star by

$$\ddot{\vec{\rho}}_\mathrm{s} = G\sum_{k=1}^{n}\frac{M_k}{r_k^3}\vec{r}_k,$$

where $\vec{r}$ denotes the heliocentric distance and $\vec{\rho}$ the distance to a fixed reference point. With $\vec{r}_i = \vec{\rho}_i - \vec{\rho}_\mathrm{s}$ one obtains

$$\begin{aligned}\ddot{\vec{r}}_i &= \ddot{\vec{\rho}}_i - \ddot{\vec{\rho}}_\mathrm{s}\\ &= -G\frac{M_\mathrm{s}}{r_i^3}\vec{r}_i - G\sum_{\substack{k=1\\k\neq i}}^{n}\frac{M_k}{|\vec{r}_i-\vec{r}_k|^3}(\vec{r}_i-\vec{r}_k) - G\sum_{k=1}^{n}\frac{M_k}{r_k^3}\vec{r}_k,\end{aligned}$$

and

$$\ddot{\vec{r}}_i + \frac{G(M_\mathrm{s}+M_i)}{r_i^3}\vec{r}_i = -G\sum_{\substack{k=1\\k\neq i}}^{n}M_k\left[\frac{1}{|\vec{r}_i-\vec{r}_k|^3}(\vec{r}_i-\vec{r}_k)+\frac{1}{r_k^3}\vec{r}_k\right].$$

This is just the sum of the contributions from the $n-1$ other planets. Similar to the one-planet case, the disturbing function for the i^th planet in the n-planet problem is the function whose gradient with respect to $\vec{r}_i$ gives the expression in the right-hand side:

$$R_i = G\sum_{\substack{k=1\\k\neq i}}^{n}M_k\left[\frac{1}{|\vec{r}_i-\vec{r}_k|} - \frac{\vec{r}_i\,\vec{r}_k}{r_k^3}\right]. \tag{1}$$

Equivalently, we could write

$$R_i = G\sum_{\substack{k=1\\k\neq i}}^{n}M_k\left[\frac{1}{\Delta_{ik}} - \frac{r_i\,\cos H_{ik}}{r_k^2}\right],$$

where $\Delta_{ik}\equiv|\vec{r}_i-\vec{r}_k|$ is the distance and $H_{ik}=\vec{r}_i\vec{r}_k/(r_ir_k)$ the stello-centric angle between planets i and k.

Problem 12.2

The disturbing function for the two-planet problem is

$$R = GM'\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{\vec{r}\,\vec{r}'}{r'^3}\right]$$

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or

$$R = GM' \left[\frac{1}{\Delta} - \frac{r \cos H}{r'^2} \right],$$

where

$$\begin{aligned} \Delta &= \left| \vec{r} - \vec{r}' \right| = \sqrt{r^2 + r'^2 - 2rr' \cos H}, && (2) \\ \cos H &= \frac{xx' + yy' + zz'}{rr'}, && (3) \end{aligned}$$

with

$$\begin{aligned} r &= \frac{a(1-e^2)}{1 + e\cos\theta}, && (4) \\ x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos I), && (5) \\ y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos I), && (6) \\ z &= r \sin u \sin I, && (7) \\ u &\equiv \omega + \theta, && (8) \end{aligned}$$

plus similar expressions for the primed quantitites.
In order to show that the expansion of R contains cos-terms only, it is sufficient to prove that R is even with respect to λ, λ', ϖ, ϖ', Ω, and Ω', i.e. that

$$R(\lambda, \lambda', \varpi, \varpi', \Omega, \Omega') = R(-\lambda, -\lambda', -\varpi, -\varpi', -\Omega, -\Omega'). \qquad (9)$$

Consider now the replacement

$$\lambda \to -\lambda, \qquad \varpi \to -\varpi, \qquad \Omega \to -\Omega.$$

Since

$$\begin{aligned} \varpi &\equiv \Omega + \omega, && (10) \\ \lambda &\equiv \Omega + \omega + M, && (11) \end{aligned}$$

it is equivalent to

$$M \to -M, \qquad \omega \to -\omega, \qquad \Omega \to -\Omega.$$

Then, the Kepler equation

$$M = E - e \sin E \qquad \text{and} \qquad \sin\theta = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}$$

lead to

$$M \to -M \quad \Longleftrightarrow \quad E \to -E \quad \Longleftrightarrow \quad \theta \to -\theta.$$

Substituting negative angles for positiv angles $(\lambda, \varpi, \Omega)$ in eqs. (4)–(8), we obtain

$$r \to r \qquad \theta \to -\theta, \qquad u \to -u,$$

and

$$x \to x, \qquad y \to -y, \qquad z \to -z.$$

Obviously, the same applies to the primed coordinates:

$$r' \to r', \qquad \theta' \to -\theta', \qquad u' \to -u',$$

and

$$x' \to x', \qquad y' \to -y', \qquad z' \to -z'.$$

From this, we find

$$\cos H \to \cos H, \qquad \Delta \to \Delta$$

and finally

$$R \to R.$$

Problem 12.3

A series expansion of the perturbing function is given by

$$R = \sum_{j_1,\dots,j_6} A_{j_1,\dots,j_6} \exp\left[i(j_1\lambda + j_2\lambda' + j_3\varpi + j_4\varpi' + j_5\Omega + j_6\Omega')\right].$$

This expansion has the d'Alembert property:

$$A_{j_1,\dots,j_6} \neq 0 \quad \Longrightarrow \quad \sum_n j_n = 0,$$

i. e. the expansion contains only terms where the sum over the j_n vanishes.
An expansion in "normal" orbital elements could be written as

$$R = \sum_{k_1,\dots,k_6} B_{k_1,\dots,k_6} \exp\left[i(k_1 M + k_2 M' + k_3\omega + k_4\omega' + k_5\Omega + k_6\Omega')\right].$$

Since $\lambda \equiv M + \omega + \Omega$ and $\varpi = \omega + \Omega$ we can connect the arguments to the exponential/cosine terms in both expansions in the following way:

$$\begin{gathered} j_1\lambda + j_2\lambda' + j_3\varpi + j_4\varpi' + j_5\Omega + j_6\Omega' \\ = \\ j_1 M + j_2 M' + (j_1 + j_3)\omega + (j_2 + j_4)\omega' + (j_1 + j_3 + j_5)\Omega + (j_2 + j_4 + j_6)\Omega' \\ = \\ k_1 M + k_2 M' + k_3\omega + k_4\omega' + k_5\Omega + k_6\Omega', \end{gathered}$$

which leads us to

$$\begin{aligned} k_1 &\equiv j_1 & k_2 &\equiv j_2 \\ k_3 &\equiv j_1 + j_3 & k_4 &\equiv j_2 + j_4 \\ k_5 &\equiv j_1 + j_3 + j_5 & k_6 &\equiv j_2 + j_4 + j_6. \end{aligned}$$

The sum in the original d'Alembert property therefore changes to

$$\sum_n j_n = j_1 + j_3 + j_5 + j_2 + j_4 + j_6 = k_5 + k_6.$$

The modified d'Alembert property is

$$B_{k_1,\dots,k_6} \neq 0 \quad \Longrightarrow \quad \underline{\underline{k_5 + k_6 = 0}} \quad \not\Longrightarrow \quad \sum_n k_n = 0.$$

The sum over all the new k_n *can* be zero, but *does not have to be* zero. Only the sum $k_5 + k_6$ is constrained.
Additional info on the background of the d'Alembert property: The reference direction in the N-body problem is an arbitrary direction, having no physical meaning. The absolute values of the longitudes of ascending nodes are measured with respect to that arbitrary reference direction, and hence, cannot appear directly in any particle–particle interaction term. As a result, terms with solitary Ω or Ω' cannot appear in any expansion either. Only relative orientations among perturbing bodies play a role, i. e. only terms with $\Omega - \Omega'$ can appear. Formally, we can write

$$k_5\Omega + k_6\Omega' \;=\; k_5\Omega \underbrace{- k_5\Omega' + k_5\Omega'}_{=0} + k_6\Omega' \;=\; \underbrace{k_5}_{\text{unconstrained}} \overbrace{(\Omega - \Omega')}^{\text{allowed}} \;+\; \underbrace{(k_5 + k_6)}_{\text{must be } = 0} \overbrace{\Omega'}^{\text{not allowed}} \;=\; k_5(\Omega - \Omega').$$

The remaining k_1 to k_4 are coefficients to the M and ω, which are measured relative to the ascending nodes and the periapses, and hence, are not arbitrary. The k_1 to k_4 (and k_5 *or* k_6, not both) can take any value. On the other hand, λ, ϖ, and Ω *all* relate to the reference direction (all contain Ω). The d'Alembert property in these angles therefore involves all six coefficients, j_1 to j_6.