Celestial Mechanics – Solutions

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Unit 11

Problem 11.1

In the lecture it was shown that

$$\langle\dot{\Omega}\rangle = -\frac{3\nu\cos I}{\kappa a^{7/2}(1-e^2)^2}$$

and

$$\langle\dot{\omega}\rangle = +\frac{6\nu\left(1-\frac{5}{4}\sin^2 I\right)}{\kappa a^{7/2}(1-e^2)^2}.$$

The resulting orbit-averaged change rate of the longitude of periapsis, $\varpi = \Omega + \omega$, is

$$\langle\dot{\varpi}\rangle = \langle\dot{\Omega}\rangle + \langle\dot{\omega}\rangle = +\frac{3\nu}{\kappa a^{7/2}(1-e^2)^2}\Big(-\cos I + 2 - \tfrac{5}{2}\underbrace{\sin^2 I}_{=1-\cos^2 I}\Big).$$

This change rate vanishes if

$$\begin{aligned}
-\cos I + 2 - \tfrac{5}{2}\left(1-\cos^2 I\right) &= 0 \\
-\tfrac{2}{5}\cos I - \tfrac{1}{5} + \cos^2 I &= 0 \qquad \text{(quadratic equation)} \\
\cos I &= \tfrac{1}{5} \pm \sqrt{\tfrac{1}{25}+\tfrac{1}{5}} \\
\cos I &= \frac{1\pm\sqrt{6}}{5} \\
I &\approx \begin{cases} 46.4^\circ \\ 106.9^\circ \end{cases}
\end{aligned}$$

Problem 11.2

From the Poynting-Robertson force

$$\vec{F}_{\mathrm{PR}} = -\frac{S_0}{r^2}\left(\frac{2\dot{r}}{c}\vec{e}_r + \frac{r\dot{\theta}}{c}\vec{e}_\theta\right),$$

we find the radial and the tangential perturbing force to be

$$S = \vec{e}_r\vec{F}_{\mathrm{PR}} = -\frac{2S_0\dot{r}}{cr^2} \qquad \text{and} \qquad T = \vec{e}_\theta\vec{F}_{\mathrm{PR}} = -\frac{S_0\dot{\theta}}{rc}, \tag{1}$$

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respectively. We now write down the Gauss perturbation equation for $\dot{a}$:

$$
\begin{aligned}
&& \dot{a} &= 2a^2[e\sin\theta\, S' + \underbrace{(1+e\cos\theta)}_{p/r} T'] \\
S' = \frac{S}{\kappa\sqrt{p}} \quad \text{and} \quad T' = \frac{T}{\kappa\sqrt{p}}: && &= \frac{2a^2}{\kappa\sqrt{p}}\left[Se\sin\theta + T\frac{p}{r}\right] \qquad (2) \\
\text{Eq. (1)}: && &= -\frac{2a^2 S_0}{c\kappa\sqrt{p}}\left[\frac{2\dot{r}}{r^2}e\sin\theta + \frac{p\dot{\theta}}{r^2}\right] \\
\dot{r} = \frac{r^2}{p}e\sin\theta\,\dot{\theta} \quad \text{and} \quad r^2\dot{\theta} = \kappa\sqrt{p}: && &= -\frac{2a^2 S_0}{c\kappa\sqrt{p}}\left[\frac{2\kappa}{\sqrt{p}r^2}e^2\sin^2\theta + \frac{\kappa p^{3/2}}{r^4}\right] \\
&& &= -\frac{2S_0}{cp}\left[\frac{2a^2}{r^2}e^2\sin^2\theta + \frac{a^2p^2}{r^4}\right] \\
\sin^2\theta = \frac{1}{2}[1-\cos(2\theta)]: && &= -\frac{2S_0}{cp}\left[e^2\frac{a^2}{r^2} - e^2\frac{a^2}{r^2}\cos(2\theta) + \frac{a^4}{r^4}(1-e^2)^2\right].
\end{aligned}
$$

The average of that change is

$$
\langle\dot{a}\rangle = -\frac{2S_0}{2\pi cp}\int_0^{2\pi}\left[\underbrace{e^2\frac{a^2}{r^2}}_{e^2X_0^{-2,0}} - \underbrace{e^2\frac{a^2}{r^2}\cos(2\theta)}_{e^2X_0^{-2,2}} + \underbrace{\frac{a^4}{r^4}\left(1-e^2\right)^2}_{X_0^{-4,0}(1-e^2)^2}\right]\mathrm{d}M,
$$

From the lecture and Problem 9.1, we know these Hansen coefficients:

$$
\begin{aligned}
X_0^{-2,0} &= \frac{1}{\sqrt{1-e^2}}, \\
X_0^{-2,2} &= 0, \\
X_0^{-4,0} &= \frac{1+e^2/2}{(1-e^2)^{5/2}},
\end{aligned}
$$

we obtain

$$
\langle\dot{a}\rangle = \underline{\underline{-\frac{S_0}{ca}\frac{2+3e^2}{(1-e^2)^{3/2}}}}.
$$

Bonus Problem

Now we can estimate S_0 to get some numbers for $\langle\dot{a}\rangle$. To this end, we consider an ideally absorbing, spherical particle with a radius s and a density ρ. It has the mass $m = (4/3)\pi\rho s^3$ and cross section $\sigma = \pi s^2$.

Using the stellar luminosity L, the radiation power that the particle intercepts from the star is

$$
P = \frac{\sigma L}{4\pi r^2}.
$$

We have

$$
\frac{S_0}{r^2} = \frac{1}{m}\frac{P}{c} = \frac{Ls^2}{4mcr^2}
$$

and we find

$$S_0 = \frac{Ls^2}{4mc},$$
$$\langle \dot{a} \rangle = -\frac{Ls^2}{4amc^2} \frac{2+3e^2}{(1-e^2)^{3/2}} = -\frac{3L}{16\pi\rho asc^2} \frac{2+3e^2}{(1-e^2)^{3/2}}.$$

This we can rewrite in typical units in the solar system:

$$\langle \dot{a} \rangle = -0.36 \frac{2+3e^2}{(1-e^2)^{3/2}} \frac{L}{L_\odot} \frac{\mathrm{g\,cm^{-3}}}{\rho} \frac{\mu\mathrm{m}}{s} \frac{\mathrm{au}}{a} \frac{\mathrm{au}}{\mathrm{kyr}} \tag{3}$$

For a particle with $s = 1\,\mu\mathrm{m}$ and $\rho = 1\,\mathrm{g\,cm^{-3}}$ in a circular orbit ($e = 0$) around the sun at a distance of 1 au, the Poynting-Robertson lifetime is given by

$$T_{\mathrm{PR}} \equiv \left| \frac{a}{\langle \dot{a} \rangle} \right| \approx 1388 \frac{L_\odot}{L} \frac{\rho}{\mathrm{g\,cm^{-3}}} \frac{s}{\mu\mathrm{m}} \left(\frac{a}{\mathrm{au}} \right)^2 \mathrm{yr} = 1388\,\mathrm{yr},$$

which means that such a particle would spiral from an Earth orbit towards the Sun in a few thousand years.

That Poynting-Robertson time is proportional to a^2 and to s. Therefore, even for $a = 30\,\mathrm{au}$ and $s = 100\,\mu\mathrm{m}$, it is “just” a few hundred million years, i.e. much less that the age of the Solar System. We conclude that the interplanetary dust particles cannot be primordial. In fact, studies show that even grains much larger than $\sim 1\,\mathrm{mm}$ cannot be primordial – they will be eliminated by another effect, namely mutual collisions.

Where do they come from? They are continuously supplied by comets and asteroids. By the way, it is still the matter of hot debate whether the cometary or the asteroidal sources are more important.