Celestial Mechanics – Solutions

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Unit 10

Problem 10.1

The Hansen coefficients are defined through

$$X_0^{n,m} \equiv \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n \cos(m\theta)\mathrm{d}M.$$

In order solve the integral, we need to bring everything to a common integration variable. The best choice can vary, depending on the problem. We start with true anomaly θ: from

$$\underbrace{r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \kappa\sqrt{a(1-e^2)}}_{\text{Kepler's 2nd}} \qquad \text{and} \qquad \underbrace{\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\kappa}{a^{3/2}}}_{\text{variant of Kepler's 3rd}},$$

we find

$$\frac{a^2}{r^2}\mathrm{d}M = \frac{\mathrm{d}\theta}{\sqrt{1-e^2}},$$

so that

$$X_0^{n,m} \equiv \frac{1}{2\pi\sqrt{1-e^2}}\int_0^{2\pi}\left(\frac{r}{a}\right)^{n+2}\cos(m\theta)\mathrm{d}\theta. \qquad (1)$$

Using

$$r = \frac{a(1-e^2)}{1+e\cos\theta},$$

Eq. (1) takes the form

$$X_0^{n,m} \equiv \frac{1}{2\pi}(1-e^2)^{n+3/2}\int_0^{2\pi}(1+e\cos\theta)^{-n-2}\cos(m\theta)\mathrm{d}\theta.$$

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The Hansen coefficients in question are are:

$$X_0^{-2,2} = \frac{1}{2\pi(1-e^2)^{1/2}} \int_0^{2\pi} \cos(2\theta)\mathrm{d}\theta \;=\; \underline{\underline{0}}$$

$$\begin{aligned}
X_0^{-3,1} &= \frac{1}{2\pi(1-e^2)^{3/2}} \int_0^{2\pi} (1+e\cos\theta)\cos\theta\;\mathrm{d}\theta \\
&= \frac{1}{2\pi(1-e^2)^{3/2}} \int_0^{2\pi} (\underbrace{\cos\theta}_{\int\ldots=0} + e\underbrace{\cos^2\theta}_{\int\ldots=\pi})\;\mathrm{d}\theta \\
&= \underline{\underline{\frac{e}{2(1-e^2)^{3/2}}}} \\
X_0^{-4,0} &= \frac{1}{2\pi(1-e^2)^{5/2}} \int_0^{2\pi} (1+e\cos\theta)^2\mathrm{d}\theta \\
&= \frac{1}{2\pi(1-e^2)^{5/2}} \int_0^{2\pi} (\underbrace{1}_{\int\ldots=2\pi} + 2e\underbrace{\cos\theta}_{\int\ldots=0} + e^2\underbrace{\cos^2\theta}_{\int\ldots=\pi})\mathrm{d}\theta \\
&= \underline{\underline{\frac{1+e^2/2}{(1-e^2)^{5/2}}}}.
\end{aligned}$$

All cosine terms are even with respect to the midpoint (π) of the integration interval. Hence, they can be rewritten as $2\int_0^{\pi}\ldots\mathrm{d}\theta$. While cos is odd with respect to the new midpoint ($\pi/2$), $\cos^2$ is even, resulting in all terms with even exponents being even (with respect to $\pi/2$) and all terms with odd exponents being odd. The integral over the odd terms from 0 to $\pi/2$ vanishes, while the corresponding integral over the even integrals may be non-zero. Partial integration can be used for these even terms.

Problem 10.2

The θ integral for $X_0^{2,0}$ is

$$\frac{(1-e^2)^{7/2}}{2\pi} \int_0^{2\pi} \frac{1}{(1+e\cos\theta)^4}\;\mathrm{d}\theta,$$

which is harder to solve directly. The clue to finding this Hansen coefficient is to make another transformation of variables: you have to go from M to E (rather than to θ as in the previous problem). Differentiating the Kepler equation, $E - e\sin E = M$, we find

$$\mathrm{d}M = (1-e\cos E)\;\mathrm{d}E = \frac{r}{a}\,\mathrm{d}E.$$

Therefore,

$$\begin{aligned}
X_0^{2,0} &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \mathrm{d}M \\
&= \frac{1}{2\pi}\int_0^{2\pi}(1-e\cos E)^3 \mathrm{d}E \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Big(\underbrace{1}_{\int\ldots=2\pi} - 3e\underbrace{\cos E}_{\int\ldots=0} + 3e^2\underbrace{\cos^2 E}_{\int\ldots=\pi} - e^3\underbrace{\cos^3 E}_{\int\ldots=0}\Big)\mathrm{d}E \\
&= \frac{1}{2\pi}\left[2\pi+3e^2\pi\right] \\
&= \underline{\underline{1+\frac{3e^2}{2}}}.
\end{aligned}$$

Problem 10.3

As in the previous problem, we rewrite

$$X_0^{n,0} \equiv \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n \mathrm{d}M$$

as

$$X_0^{n,0} = \frac{1}{2\pi}\int_0^{2\pi}(1-e\cos E)^{n+1}\mathrm{d}E.$$

Using the binomial expansion,

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (0 \le k \le n)$$

are the binomial coefficients and $a=1$, $b=-e\cos E$, we find

$$X_0^{n,0} = \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\binom{n+1}{k}(-e\cos E)^k \mathrm{d}E.$$

By including all terms with $k \le 2$, we obtain e^2 accuracy:

$$\begin{aligned}
X_0^{n,0} &= \frac{1}{2\pi}\int_0^{2\pi}\left[\underbrace{\binom{n+1}{0}}_{=1} - \binom{n+1}{1}e\underbrace{\cos E}_{\int\ldots=0} + \underbrace{\binom{n+1}{2}}_{=\frac{n(n+1)}{2}}e^2\underbrace{\cos^2 E}_{\int\ldots=\pi}\right]\mathrm{d}E + o(e^2) \\
&= \underline{\underline{1+\frac{n(n+1)}{4}e^2}} + o(e^2).
\end{aligned}$$

For $n=2$, the $o(e^2)$ terms vanish and the result agrees with the solution of the previous problem.

Alternatively, we could have expanded $f(e) = (1 - e\cos E)^{n+1}$ in a Taylor series around $e = 0$:

$$f(e) = f(0) + ef'(0) + \frac{e^2}{2} f''(0) + o(e^2) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} e^k,$$

where

$$\begin{aligned}
f'(e) &= -(n+1)(1-e\cos E)^n \cos E, & f'(0) &= -(n+1)\cos E \\
f''(e) &= n(n+1)(1-e\cos E)^{n-1}\cos^2 E, & f''(0) &= n(n+1)\cos^2 E \\
&\vdots & &\vdots \\
f^{(k)}(e) &= \frac{(n+1)!}{(n+1-k)!}(1-e\cos E)^{n+1-k}(-\cos E)^k, & f^{(k)}(0) &= \frac{(n+1)!}{(n+1-k)!}(-\cos E)^k,
\end{aligned}$$

with $k \leq n+1$, such that

$$\begin{aligned}
f(e) &= 1 - e(n+1)\cos E + \frac{n(n+1)}{2} e^2 \cos^2 E + o(e^2) \\
&= \sum_{k=0}^{n+1} \frac{(n+1)!}{(n+1-k)!\,k!}(-e\cos E)^k \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} (-e\cos E)^k.
\end{aligned}$$

Eventually, we are back at

$$X_0^{n,0} = \frac{1}{2\pi}\int_0^{2\pi} \left[1 - e(n+1)\cos E + \frac{n(n+1)}{2} e^2 \cos^2 E + o(e^2)\right] \mathrm{d}E = 1 + \frac{n(n+1)}{4} e^2 + o(e^2).$$