Celestial Mechanics – Solutions

Alexander V. Krivov & Tobias Stein[1]

Unit 9

Problem 9.1

(a) The Delaunay-Hamiltonian is given by

$$\mathscr{H} = \frac{\kappa^4}{2L_1^2} + R,$$

and since we perform a contact transformation, the new Poincaré-Hamlitonian $\tilde{\mathscr{H}}$ must be the same. However, we know that $L_1 \equiv \tilde{L}_1$, and thus,

$$\tilde{\mathscr{H}} = \frac{\kappa^4}{2\tilde{L}_1^2} + R.$$

Now we can *construct/choose* the momenta $\tilde{l}_k$ such that we have a contact transformation. Inserting

$$\tilde{L}_1 = L_1, \qquad \tilde{L}_2 = L_1 - L_2, \qquad \tilde{L}_3 = L_2 - L_3$$

into

$$\sum_k (l_k \mathrm{d}L_k - \tilde{l}_k \mathrm{d}\tilde{L}_k) = 0,$$

and expanding the sum, we find

$$\begin{aligned} 0 &= l_1 \mathrm{d}L_1 - \tilde{l}_1 \mathrm{d}L_1 \\ &+ l_2 \mathrm{d}L_2 - \tilde{l}_2 \mathrm{d}(L_1 - L_2) \\ &+ l_3 \mathrm{d}L_3 - \tilde{l}_3 \mathrm{d}(L_2 - L_3). \end{aligned}$$

Now, we sort the differentials and obtain

$$\begin{aligned} 0 &= (l_1 - \tilde{l}_1 - \tilde{l}_2)\mathrm{d}L_1 \\ &+ (l_2 + \tilde{l}_2 - \tilde{l}_3)\mathrm{d}L_2 \\ &+ (l_3 + \tilde{l}_3)\mathrm{d}L_3. \end{aligned}$$

Since the L_k are pairwise independent, all the coefficients to the differentials have to vanish:

$$\begin{aligned} 0 &= l_1 - \tilde{l}_1 - \tilde{l}_2, \\ 0 &= l_2 + \tilde{l}_2 - \tilde{l}_3, \\ 0 &= l_3 + \tilde{l}_3. \end{aligned}$$

Solving this set of equations from bottom to top, the result is

$$\begin{aligned} \tilde{l}_3 &= -l_3, \\ \tilde{l}_2 &= -l_2 - l_3, \\ \tilde{l}_1 &= l_1 + l_2 + l_3. \end{aligned}$$

[1] tobias.stein@uni-jena.de

Using the known Delaunay variables

$$\begin{aligned} L_1 &= \kappa\sqrt{a}, & l_1 &= M, \\ L_2 &= \kappa\sqrt{a(1-e^2)}, & l_2 &= \omega, \\ L_3 &= \kappa\sqrt{a(1-e^2)}\cos I, & l_3 &= \Omega, \end{aligned}$$

we finally find the Poincaré variables

$$\begin{aligned} \tilde{L}_1 &= \kappa\sqrt{a}, & \tilde{l}_1 &= M+\omega+\Omega, \\ \tilde{L}_2 &= \kappa\sqrt{a}\left(1-\sqrt{1-e^2}\right), & \tilde{l}_2 &= -\omega-\Omega, \\ \tilde{L}_3 &= \kappa\sqrt{a(1-e^2)}(1-\cos I), & \tilde{l}_3 &= -\Omega. \end{aligned}$$

(b) Planetary orbits in the solar system are almost circular (small e) and almost co-planar (small I). The advantage of the Poincaré variables is that they are more sensitive to small changes in e and I because

$$\begin{aligned} &\tilde{L}_2 \to \frac{\kappa}{2}\sqrt{a}e^2 \to 0 && \text{for } e \to 0, \\ &\tilde{L}_3 \to \frac{\kappa}{2}\sqrt{a(1-e^2)}I^2 \to 0 && \text{for } I \to 0. \end{aligned}$$

The corresponding errors are

$$\Delta\tilde{L}_2 \to \kappa\sqrt{a}\, e\Delta e \qquad \text{or} \qquad \frac{\Delta\tilde{L}_2}{\tilde{L}_2} \to 2\frac{\Delta e}{e}$$

and

$$\Delta\tilde{L}_3 \to \kappa\sqrt{a(1-e^2)}\, I\Delta I \qquad \text{or} \qquad \frac{\Delta\tilde{L}_3}{\tilde{L}_3} \to 2\frac{\Delta I}{I}.$$

In contrast, for the Delaunay variables, we have

$$\begin{aligned} &L_2 \to \kappa\sqrt{a}\left(1-\frac{e^2}{2}\right) \approx \text{const} && \text{for } e \to 0, \\ &L_3 \to \kappa\sqrt{a(1-e^2)}\left(1-\frac{I^2}{2}\right) \approx \text{const} && \text{for } I \to 0. \end{aligned}$$

and

$$\begin{aligned} &\Delta L_2 \to \kappa\sqrt{a}\, e\Delta e && \text{or} && \frac{\Delta L_2}{L_2} \to \frac{e\Delta e}{1-\frac{e^2}{2}} \to e^2\frac{\Delta e}{e} \to 0 && \text{or} && \frac{\Delta e}{e} = \frac{1}{e^2}\frac{\Delta L_2}{L_2} \to \infty \\ &\Delta L_3 \to \kappa\sqrt{a(1-e^2)}\, I\Delta I && \text{or} && \frac{\Delta L_3}{L_3} \to I^2\frac{\Delta I}{I} \to 0 && \text{or} && \frac{\Delta I}{I} = \frac{1}{I^2}\frac{\Delta L_3}{L_3} \to \infty, \end{aligned}$$

meaning that these variables are insensitive to small relative changes in e and I for almost circular, non-inclined orbits. In turn, this is a problem for numerical calculations, where small errors in L_2 and L_3 (such as rounding errors) can lead to huge errors in e and I.