Celestial Mechanics – Solutions

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Unit 8

Problem 8.1

The semilatus rectum is defined through

$$p \equiv a(1-e^2),$$

and accordingly, its first derivative with respect to time is

$$\dot{p} = \dot{a}(1-e^2) - 2ae\dot{e} = \frac{\dot{a}p}{a} - 2ae\dot{e}.$$

(a) Using the known Gauss perturbation equations for $\dot{a}$ and $\dot{e}$,

$$\begin{aligned} \dot{a} &= 2a^2\left[e\sin\theta\, S' + (1+e\cos\theta)\, T'\right], \\ \dot{e} &= p\sin\theta\, S' + p\cdot(\cos\theta + \cos E)T', \end{aligned}$$

we find

$$\dot{p} = 2ap(1-e\cos E)T' = 2prT'.$$

(b) The Lagrange perturbation equations for p can be derived from

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{\partial R}{\partial M}, \\ \dot{e} &= \underbrace{\frac{1-e^2}{ena^2}\frac{\partial R}{\partial M}}_{=\frac{\dot{a}p}{2ea^2}} - \frac{\sqrt{1-e^2}}{ena^2}\frac{\partial R}{\partial \omega}. \end{aligned} \qquad (1)$$

We obtain

$$\dot{p} = \underbrace{\frac{\dot{a}p}{a} - \frac{\dot{a}p}{a}}_{=0} + 2\frac{\sqrt{1-e^2}}{na}\frac{\partial R}{\partial \omega} = 2\frac{\sqrt{p}}{\kappa}\frac{\partial R}{\partial \omega}. \qquad (2)$$

Problem 8.2

Since the orbital motion is assumed to be circular, we can neglect the radial part of the drag force, S, and the Gauss perturbation equation for the semimajor axis reduces to

$$\dot{a} = 2a^2T',$$

where

$$T' = \frac{T}{\kappa\sqrt{p}},$$

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and the azimuthal part of the *specific* drag force, T, is given by

$$T = \ddot{x} = -\frac{C_\mathrm{d}\sigma\rho\dot{x}^2}{2m}.$$

Therefore,

$$\dot{a} = -a^2\frac{C_\mathrm{d}\sigma\rho\dot{x}^2}{m\kappa\sqrt{a}},$$

where we used the fact that $p = a$ for $e = 0$. In addition, we know that the velocity $\dot{x}$ equals the Keplerian velocity:

$$\dot{x} = \frac{\kappa}{\sqrt{a}}.$$

We obtain

$$\dot{a} = -\frac{C_\mathrm{d}\sigma\rho\kappa\sqrt{a}}{m} = -\frac{C_\mathrm{d}\sigma\rho\sqrt{GM_\oplus a}}{m}. \tag{3}$$

We know that $a = 400\,\mathrm{km} + 6373\,\mathrm{km} \approx 7\times\ 10^8\,\mathrm{cm}$, $\rho \approx 10^{-15}\,\mathrm{g\,cm^{-3}}$, and $\kappa^2 = GM_\oplus = 4\times10^{20}\,\mathrm{cm^3\,s^{-2}}$, and we can estimate $C_\mathrm{d} \sim 1$, $\sigma \sim 30\times30\,\mathrm{m}^2 \approx 10^7\,\mathrm{cm}^2$, and $m \sim 400\,\mathrm{t} \approx 4\times10^8\,\mathrm{g}$. (Don't worry if you took, say, $\sigma = 100\,\mathrm{m}^2$ and $m = 100\,\mathrm{t}$ – it's fine for a rough estimate.) Substituting the numbers into Eq. (3), we find

$$\dot{a} \approx -0.01\ \mathrm{cm\,s^{-1}},$$

which translates to an altitude loss of roughly 10 meters per day or about 300 meters per month.
Figure 1 shows the real evolution of the ISS's (somewhat higher) altitude, with the height loss (between the reboosts) in 2022 being (very roughly) consistent with our estimate. Our estimate for the cross section σ is likely the dominant source of error.
From our estimate, it follows that if the air density were by two orders of magnitude higher ($\rho \sim 10^{-12}\,\mathrm{g\,cm^{-3}}$), the loss rate would be on the order of a few hundred kilometers per month — comparable with the altitude itself. Obviously, space flight at atmospheric layers that thick would no longer be possible. Figure 2 shows the output of a commonly used average model for Earth's atmosphere (Source: `https://kauai.ccmc.gsfc.nasa.gov/instantrun/msis`). The critical density of $\rho \approx 10^{-12}\,\mathrm{g\,cm^{-3}}$ discussed above is reached at an altitude of $\approx 170\,\mathrm{km}$. There are indeed no satellites flying closer to the Earth surface than that.
Figure 2 shows that the density in the upper atmosphere has strong diurnal and annual variations caused by the varying solar irradiation. The 11-year solar activity cycle leads to the additional long-term variations by a factor of 10 seen in Fig. 3.

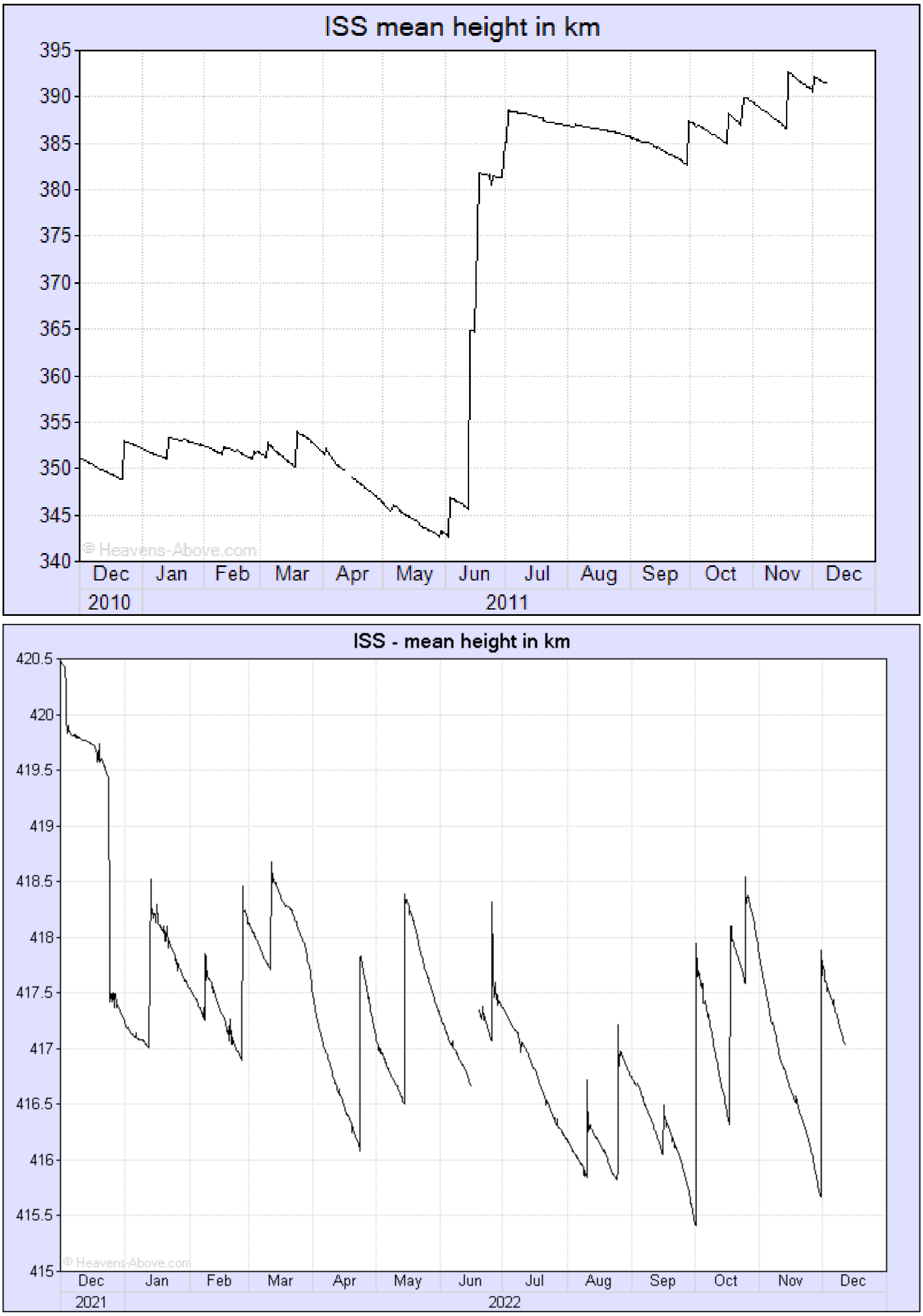

Figure 1: "This plot shows the orbital height of the ISS over one year. Clearly visible are the re-boosts which suddenly increase the height, and the gradual decay in between. The height is averaged over one orbit, and the gradual decrease is caused by atmospheric drag. As can be seen from the plot, the rate of descent is not constant and this variation is caused by changes in the density of the tenuous outer atmosphere due mainly to solar activity." (Source: `www.heavens-above.com`, top: 2011-12-07, bottom: 2022-12-12)

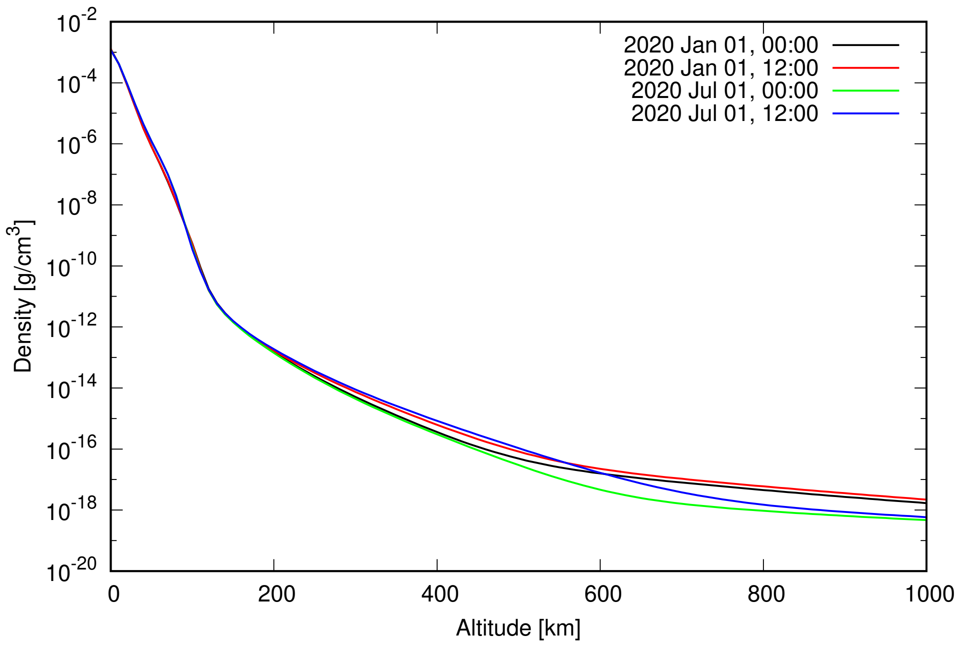

Figure 2: Average mass density of the atmosphere over altitude for winter/summer and day/night at moderate latitudes.

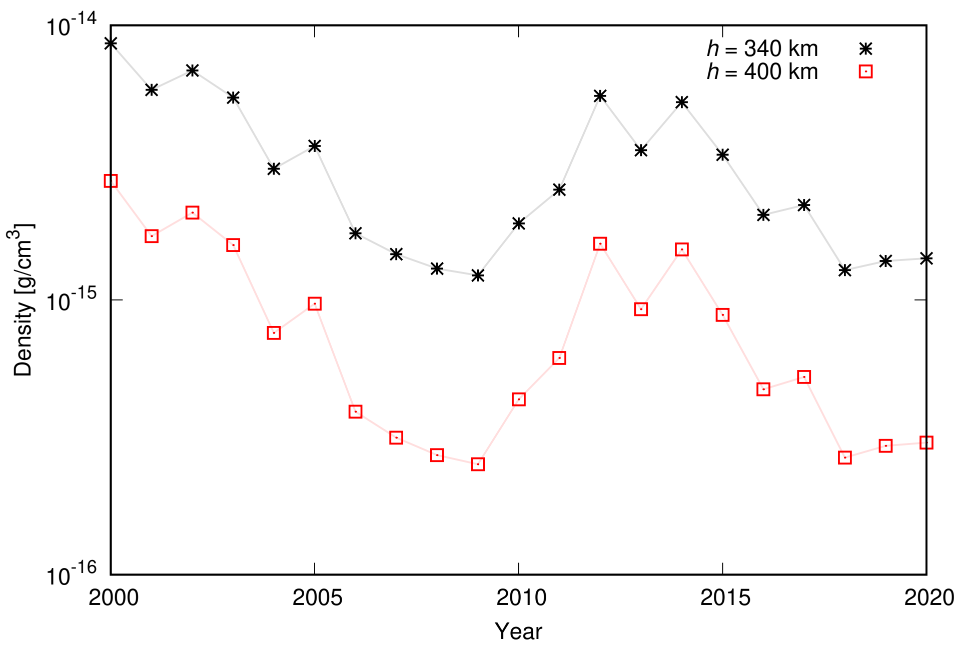

Figure 3: Atmospheric mass densities in summer nights at altitudes of 340 km and 400 km between the years 2000 to 2020 at moderate latitudes: strong correlation with solar activity.