Celestial Mechanics – Solutions

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Unit 4

Problem 4.1

Eccentric and mean anomaly are linked through Kepler's equation:

$$M = E - e \sin E. \tag{1}$$

As explained in the lecture, the simplest way to solve the equation for E is (fixed-point) iterations: solve for (one of the occurrences of) E,

$$E = M + e \sin E, \tag{2}$$

insert the previous value on the right-hand side and obtain a new value on the left-hand side:

$$E_{n+1} = M + e \sin E_n, \tag{3}$$

As long as $e < 1$, the iterations converge, although rather slow for $M \sim \pi$. Listing 1 presents a trivial C program that performs those iterations, and Listing 2 contains the results. In some 15 iterations, the accuracy is about 1 degree. The accuracy of 0.01° requires some 30 iterations. For $M = 120°$ and $e = 0.9$, the resulting eccentric anomaly is

$$E \approx 147.62°.$$

As an alternative, we could use Newton's method, which uses the slope of a function and iterates

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \tag{4}$$

to find the roots of function $f(x)$. In our case, f and its roots are given by

$$f(E) = E - e \sin E - M = 0 \tag{5}$$

and its derivative by

$$f'(E) = \frac{\mathrm{d}f}{\mathrm{d}E} = 1 - e \cos E. \tag{6}$$

If we replace line 16 in Listing 1 by the one in Listing 3, we find that Newton's method converges more quickly (see Listing 4), which is the case for most combinations of e and M. However, for $e \gtrsim 0.87$ and $M \lesssim 30°$, Newton's method converges poorly or only by chance and the fixed-point iteration should be preferred. Figure 1 illustrates the required number of iterations for the two methods. The secant method (or "regula falsi") converges well under all circumstances, but requires more lines of code.

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Listing 1: A simple C program to solve Kepler's equation

```
#include <stdio.h>
#include <math.h>

main()
{
  double degree = atan(1.0)/45.0;
  double e, M, E, E1;

  e = 0.9;
  M = 120.0*degree;

  E1 = M;
  do
  {
    E  = E1;
    E1 = M + e*sin(E);
    printf("%10.5f\n", E1/degree);

  } while (fabs(E1 - E) > 1.0e-3*degree);

}
```

Listing 2: The output of that C program

```
164.65764
133.64370
157.31566
139.88671
153.22416
143.23063
150.86729
145.10419
149.50030
146.17159
148.70729
146.78402
148.24779
147.13654
147.98182
147.33980
147.82799
147.45708
147.73907
147.52479
147.68768
147.56389
147.65798
147.58648
147.64083
147.59952
147.63092
147.60705
147.62519
147.61141
147.62188
147.61392
147.61997
147.61537
147.61887
147.61621
147.61823
147.61670
147.61786
147.61698
```

Listing 3: A simple C program to solve Kepler's equation with Newton's method

```
#include <stdio.h>
#include <math.h>

main()
{
  double degree = atan(1.0)/45.0;
  double e, M, E, E1;

  e = 0.9;
  M = 120.0*degree;

  E1 = M;
  do
  {
    E  = E1;
    E1 = E - (E - M - e*sin(E))/(1.0 - e*cos(E));
    printf("%10.5f \n", E1/degree);
  }
  while (fabs(E1 - E) > 1.0e-3*degree);
}
```

Listing 4: The output with Newton's method

```
150.79837
147.63979
147.61736
147.61736
```

The true anomaly θ can be found from two known formulas for the radial distance r:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad \text{and} \qquad r = a(1-e\cos E),$$

whence

$$\cos\theta = \frac{1}{e}\left[\frac{(1-e^2)}{1-e\cos E} - 1\right] = \frac{\cos E - e}{1-e\cos E}\,.$$

The resulting true anomaly is between 0° and 180°, because so is the eccentric anomaly:

$$\theta \approx 172.37^\circ.$$

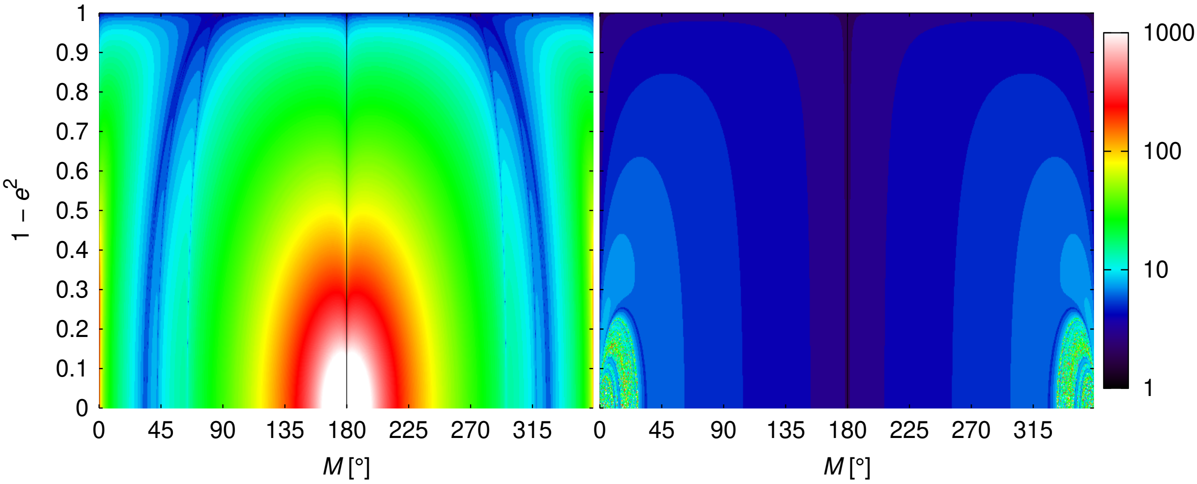

Figure 1: Required number of steps to reach an accuracy of 1×10^{-4} rad when solving Kepler's equation for E as a function of e and M. Two methods are compared: (left) fixed-point iteration, (right) Newton's method.

By the way, the relation between E and θ can be simplified by writing

$$1-\cos\theta = \frac{(1+e)(1-\cos E)}{1-e\cos E} \qquad\qquad 1+\cos\theta = \frac{(1-e)(1+\cos E)}{1-e\cos E}$$

and using the double angle formulae, these equations can be written as

$$\sin^2\frac{\theta}{2} = \frac{(1+e)}{1-e\cos E}\sin^2\frac{E}{2}\,, \qquad\qquad \cos^2\frac{\theta}{2} = \frac{(1-e)}{1-e\cos E}\cos^2\frac{E}{2}\,.$$

Hence, the relation is

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\,.$$

Problem 4.2

For illustrative purposes, Let's start with a fiducial comet with $e = 0.7$, $I = 45°$, $\Omega = 60°$, $\omega = 70°$, and $\phi = 135°$. Figure 2a shows the initial ellipse in the reference plane, with its perihelion orientated towards the reference direction, arbitrarily chosen to coincide with the x axis. In the solar system, the plane is the ecliptic plane and the direction is towards vernal equinox (♈). The red line depicts the line of node and the green line is the major axis of the ellipse. The thin black arrow shows the position vector of the comet. In Fig. 2b the ellipse was rotated by the longitude of ascending node Ω with respect to the reference direction and by the argument of pericenter ω with respect to the line of node. The last step (Fig. 2c) is to rotate the ellipse around the line of node to apply the inclination. We can apply the same steps for the actual comets Halley (Fig. 4) and Hale–Bopp (Fig. 3).
Fig. 5 shows the orbits of the comets and Earth to scale.

Bonus: we are interested in the time required to move from the current position to the aphelion,

$$\Delta t \equiv t(180°) - t(\theta_{\text{current}}). \tag{7}$$

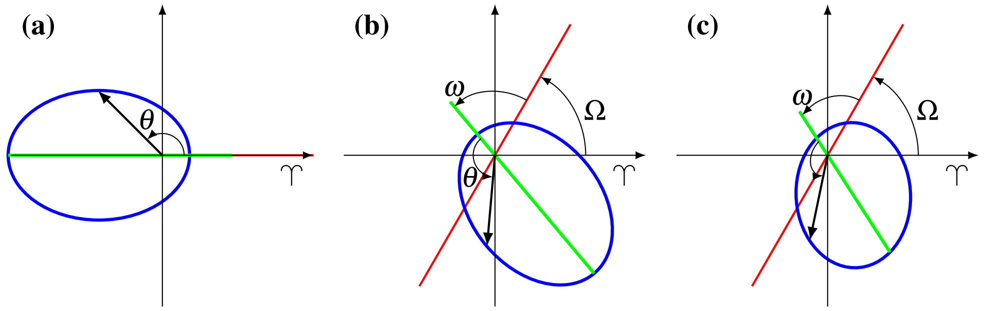

Figure 2: Sketching the orbit of a fiducial comet with $e = 0.7$, $I = 45°$, $\Omega = 60°$, $\omega = 70°$, $\theta = 135°$. **(a)** Initial position of the ellipse. **(b)** Rotate orbit by ω and Ω. **(c)** Rotate around the line of nodes to apply inclination I.

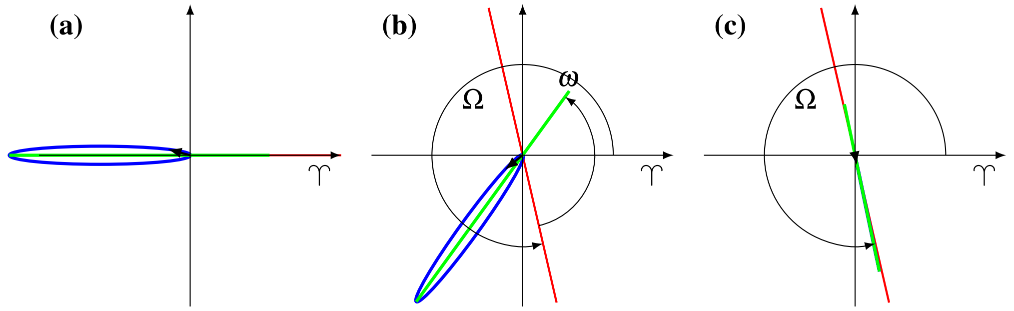

Figure 3: Sketching the orbit of comet C/1995 O1 Hale-Bopp. **(a)** Initial position of the ellipse. **(b)** Rotate orbit by ω and Ω. **(c)** Rotate around the line of nodes to apply inclination $I = 89°$.

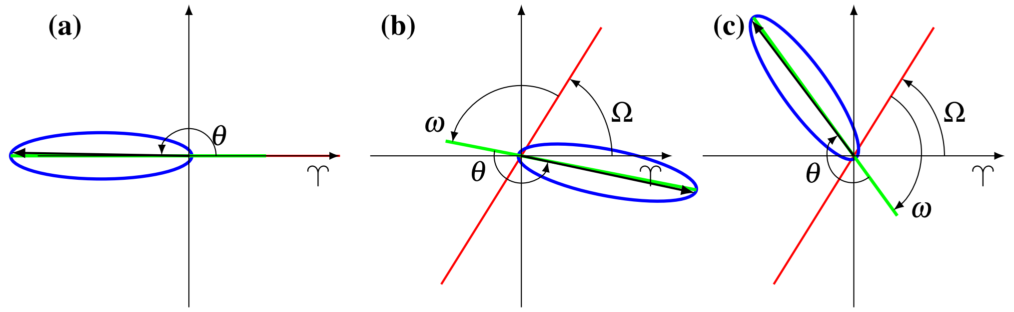

Figure 4: Sketching the orbit of comet 1P/Halley. **(a)** Initial position of the ellipse. **(b)** Rotate orbit by ω and Ω. **(c)** Flip around the line of nodes to apply inclination $I = 162°$.

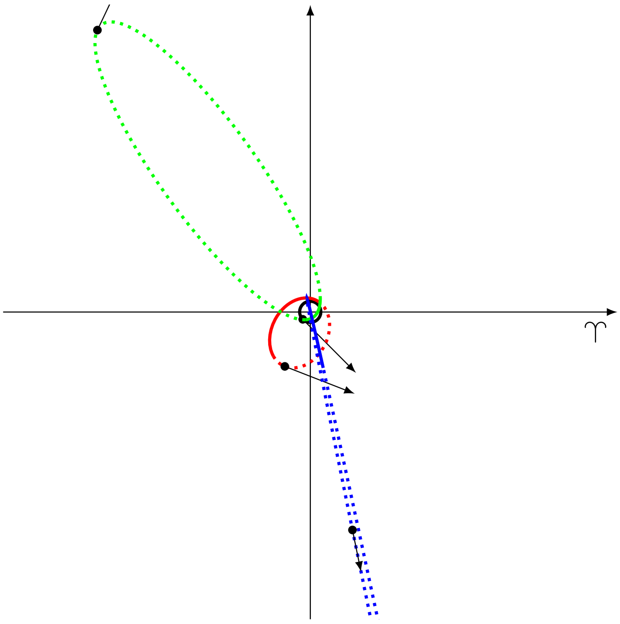

Figure 5: The orbits of comets (red) Churyumov-Gerasimenko, (green) Halley, and (blue) Hale-Bopp projected onto the ecliptic plane. Earth's orbit is plotted in black. Solid arcs are above the ecliptic plane, dotted arcs are below the ecliptic.

The time is related to mean anomaly via

$$\begin{aligned} M &= \frac{2\pi}{P}(t-T) \\ \frac{M}{\pi}P+T &= t \\ \frac{\overbrace{M(\theta=180^\circ)}^{=180^\circ}-M(\theta_{\text{current}})}{2\pi}P &= \Delta t, \end{aligned} \tag{8}$$

where T is the time of last perihelion passage and $P = 2\pi\sqrt{a^3/(GM_{\text{Sun}})}$ the orbital period. The mean anomaly M us related to the eccentric anomaly E via Kepler's equation,

$$M = E - e\sin E, \tag{9}$$

which in turn depends on the true anomaly θ via the two equations for a conic section:

$$\begin{aligned} r = \frac{a(1-e^2)}{1+e\cos\theta} &= a(1-e\cos E) \\ 1-\frac{1-e^2}{1+e\cos\theta} &= e\cos E \\ \arccos\left[\frac{1}{e}\left(1-\frac{1-e^2}{1+e\cos\theta}\right)\right] &= E \\ \arccos\left[\frac{e+\cos\theta}{1+e\cos\theta}\right] &= E. \end{aligned} \tag{10}$$

From the values given for comet Halley, $e = 0.968$, $a = 17.9$ au, $\theta_{\text{current}} = 179.97°$, we obtain

$$\begin{aligned} P &= 75.7 \text{ yr}, && (11)\\ E(\theta_{\text{current}}) &= 179.765°, && (12)\\ M(E(\theta_{\text{current}})) &= 179.537°, && (13)\\ \Delta t &= 0.097 \text{ yr} \quad \approx 0.1 \text{ yr}. && (14) \end{aligned}$$

Halley will reach its aphelion in late 2023. For Hale-Bopp's $e = 0.995$, $a = 177$ au, $\theta_{\text{current}} = 165°$, we obtain

$$\begin{aligned} P &= 2355 \text{ yr}, && (15)\\ E(\theta_{\text{current}}) &= 41.6°, && (16)\\ M(E(\theta_{\text{current}})) &= 3.76°, && (17)\\ \Delta t &\approx 1150 \text{ yr}. && (18) \end{aligned}$$

While Hale-Bopp's true anomaly appears already close to its aphelion value, its eccentric and mean anomaly are still far from aphelion. The comet's last *perihelion* passage actually happened in 1997, only some 25 years ago.

Extra info: for M (and E and θ) close to 180° we can define $M' \equiv \pi - M$ and $E' = \pi - E$ and approximate Kepler's equation:

$$\begin{aligned} \pi - M &= \pi - E + e \sin E \\ M' &= E' + e \sin E' \\ &= E' + e\left[E' + \mathcal{O}(E'^3)\right] \\ &\approx (1+e)E' && (19) \end{aligned}$$

such that

$$M \approx \pi - (1+e)(\pi - E) = E - e(\pi - E). \tag{20}$$

For $e \approx 1$, we find

$$(\pi - M) \approx 2(\pi - E). \tag{21}$$

In a similar fashion, we can let $\theta' \equiv \pi - \theta$ to find

$$\cos\theta = \cos(\pi - \theta') = -\cos\theta' \approx -1 + \frac{\theta'^2}{2}, \tag{22}$$

and hence,

$$E = \arccos\left[\frac{e - 1 + \theta'^2/2}{1 - e + e\theta'^2/2}\right] = \arccos\left[-1 + \frac{(1+e)\theta'^2}{2(1 - e + e\theta'^2/2)}\right]. \tag{23}$$

Eq. (22) can be rewritten to

$$\arccos\left[-1 + \frac{\theta'^2}{2}\right] \approx \pi - \theta' \quad \text{or} \quad \arccos[-1 + x] \approx \pi - \sqrt{2x}, \tag{24}$$

which leads us to

$$(\pi - E) \approx \sqrt{\frac{(1+e)\theta'^2}{1 - e + e\theta'^2/2}} = \theta'\sqrt{\frac{1+e}{1 - e + e\theta'^2/2}} \approx (\pi - \theta)\sqrt{\frac{2}{1-e}}, \tag{25}$$

where the last approximation assumes $\theta' \ll 1 - e$, such that

$$(\pi - M) \approx 2(\pi - \theta)\sqrt{\frac{2}{1-e}}. \tag{26}$$

Problem 4.3

Collision occurs if the distance becomes zero:

$$r(E) = a(1 - e\cos E) = 0 \tag{27}$$

Obviously, in the elliptic case ($0 \leq e < 1$ and $a > 0$), there is no real solution to that problem because an ellipse never goes through its foci. However, we can find a solution for $E(t)$ for complex instants of time. Equation (27) has the form

$$\cos E = b \qquad \text{with} \qquad b \equiv 1/e. \tag{28}$$

Noting that

$$\mathrm{e}^{iE} = \cos E + i\sin E,$$

we find that

$$\cos E = \frac{\mathrm{e}^{iE} + \mathrm{e}^{-iE}}{2}$$

or

$$2\cos E = x + \frac{1}{x},$$

where we have put $x \equiv \mathrm{e}^{iE}$. Equation (28) becomes a quadratic:

$$x^2 - 2bx + 1 = 0$$

and has two roots:

$$x = b \pm \sqrt{b^2 - 1}.$$

or

$$\mathrm{e}^{iE} = b \pm \sqrt{b^2 - 1}.$$

or

$$iE = i \cdot 2\pi k + \ln\left(b \pm \sqrt{b^2 - 1}\right), \tag{29}$$

the first term comes from 2π-periodicity of the exponential function e^{iE}. Then,

$$E = 2\pi k - i\ln\left(b \pm \sqrt{b^2 - 1}\right),$$

Since

$$b - \sqrt{b^2 - 1} = \left[b + \sqrt{b^2 - 1}\right]^{-1},$$

we can re-write our result as

$$E = 2\pi k \mp i\ln\left(b + \sqrt{b^2 - 1}\right)$$

or, remembering that $b = 1/e$,

$$E = 2\pi k \mp i\ln\left(\frac{1}{e} + \sqrt{\frac{1}{e^2} - 1}\right). \tag{30}$$

The moments of collisions can be found from Kepler's equation

$$E - e\sin E = n(t - T) \tag{31}$$

with n being the mean motion. We need $\sin E$, which is easy to calculate from Eq. (28)

$$\sin E = \mp\sqrt{1 - \cos^2 E} = \mp\sqrt{1 - \frac{1}{e^2}} = \mp\frac{1}{e}\sqrt{e^2 - 1} = \mp\frac{i}{e}\sqrt{1 - e^2}. \tag{32}$$

Now we substitute E from (30) and $\sin E$ from (32) into Kepler's equation (31). With the correct combination of the signs (!) we obtain

$$2\pi k \mp i \ln\left(\frac{1}{e}+\sqrt{\frac{1}{e^2}-1}\right) \pm i\sqrt{1-e^2} = n(t-T),$$

so that collisions take place at the time instants

$$t = T + \frac{1}{n}\left[2\pi k \mp i \ln\left(\frac{1}{e}+\sqrt{\frac{1}{e^2}-1}\right) \pm i\sqrt{1-e^2}\right],$$

or, in a compact form,

$$t = T + k\,P \mp \frac{i\beta}{n},$$

where $P = 2\pi/n$ is the orbital period and we have denoted

$$\begin{aligned}\beta &\equiv \ln\left(\frac{1}{e}+\sqrt{\frac{1}{e^2}-1}\right) - \sqrt{1-e^2} \\ &= \ln\left(1+\sqrt{1-e^2}\right) - \sqrt{1-e^2} - \ln e.\end{aligned}$$

It is easy to see that β is real and positive and $\beta \to \infty$ for $e \to 0$.

The complex instants of collisions are plotted in Fig. 6.

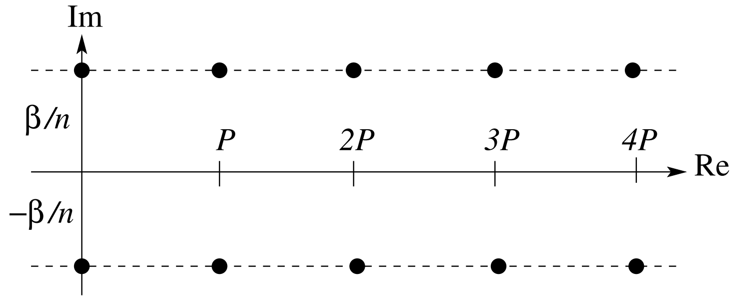

Figure 6: Instants of time (bold dots) in a complex plane at which collisions in the elliptic motion occur. Here, we have set $T = 0$ (i.e., $t = 0 + 0i$ corresponds to a pericenter passage).

Note: This exercise has practical use: the convergence radius of a Taylor series expansion for the approximation of a trajectory – e. g. for an asteroid – is equal to the minimum "temporal distance" to the next singularity in the imaginary plane. Thus, an approximation over a time longer than this minimum distance would diverge.