Celestial Mechanics – Solutions

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Unit 3

Problem 3.1

(a) The semimajor axes of the two planets can be obtained from Kepler's 3rd law:

$$\frac{a}{\mathrm{au}} = \left(\frac{P}{\mathrm{yr}}\right)^{2/3},$$

giving

$$a_\mathrm{N} = 165^{2/3}\,\mathrm{au} = 30.1\,\mathrm{au} \qquad \text{and} \qquad a_\mathrm{P} = 248^{2/3}\,\mathrm{au} = 39.5\,\mathrm{au}.$$

The minimum and the maximum distance on an elliptic orbit can be obtained from the equation that describes the orbit,

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta},$$

and are given by

$$r_\mathrm{min} = r(0) = a(1-e), \qquad r_\mathrm{max} = r(\pi) = a(1+e).$$

Thus, for Neptune's circular orbit we have

$$a_\mathrm{N} = r_\mathrm{N,min} = r_\mathrm{N,max} = 30.1\,\mathrm{au},$$

whereas for Pluto's eccentric orbit

$$r_\mathrm{P,min} = 29.6\,\mathrm{au}, \qquad r_\mathrm{P,max} = 49.3\,\mathrm{au}.$$

(b) So, we find that $r_\mathrm{P,min} < r_\mathrm{N,max}$ while $r_\mathrm{P,max} > r_\mathrm{N,min}$. The two orbits cross each other in a 2D projection onto the ecliptic plane.
However, Pluto's real orbit is inclined by 17° (Neptune: 1.7°) and the two orbits do not cross each other in three dimensions. (Extra info: Even if they did, Pluto would not undergo close encounters because his orbital motion is locked in Neptune's outer 3:2 resonance. Thus Pluto is "safe".)

(c) Kepler's 2nd law, angular momentum conservation, helps us here. We know from class that the angular momentum constant is given by

$$c = r^2\dot{\theta}. \tag{1}$$

In addition we know from the energy integral,

$$\frac{v^2}{2} - \frac{\kappa^2}{r} = \frac{h}{2} \qquad \text{and} \qquad \frac{v^2}{2} = \frac{h}{2} + \frac{\kappa^2}{r} \tag{2}$$

that the orbital velocity, v, is minimal for maximum distance, r_max, and vice versa, v is maximal for r_min. At these two locations, the apocenter and the pericenter, the radial velocity vanishes, $\dot{r} = 0$, and we find

$$v(r = r_\mathrm{min/max}) = r\dot{\theta} = \frac{c}{r}. \tag{3}$$

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Thus, the ratio of the extremal velocities is

$$\frac{v_{\mathrm{max}}}{v_{\mathrm{min}}} = \frac{v(r_{\mathrm{min}})}{v(r_{\mathrm{max}})} = \frac{r_{\mathrm{max}}}{r_{\mathrm{min}}} = \frac{1+e}{1-e} \approx \frac{5}{3}. \tag{4}$$

Extra info: if we were interested in the absolute values of v_{min} and v_{max} we could use Kepler's 2nd law to express the area of the orbital ellipse (see lecture),

$$\frac{c}{2}P = \pi ab, \tag{5}$$

with the orbital period $P = 2\pi\sqrt{a^3/\kappa^2}$ and the semi-minor axis $b = a\sqrt{1-e^2}$, we find

$$c = \frac{2\pi ab}{P} = \sqrt{a\kappa^2(1-e^2)}, \tag{6}$$

and hence,

$$v_{\mathrm{max/min}} = \frac{2\pi a^2\sqrt{1-e^2}}{Pr_{\mathrm{min/max}}}. \tag{7}$$

With $r_{\mathrm{min}} = a(1-e)$ and $r_{\mathrm{max}} = a(1+e)$, we end up at $v_{\mathrm{max}} = 6.1$ km/s and $v_{\mathrm{min}} = 3.7$ km/s.

Problem 3.2

We know from the lectures that the semi-major axis a of an orbit is linked with the constant total energy:

$$\frac{v^2}{2} - \frac{\kappa^2}{r} = \frac{h}{2} = -\frac{\kappa^2}{2a}, \tag{8}$$

where v is the launch velocity, r the asteroid's radius, and $\kappa^2 = G(M+m) \approx GM$, with M the mass of the asteroid and m the mass of the stone. Solving for a and introducing the escape velocity $v_{\mathrm{esc}}^2 \equiv 2\kappa^2/r$, we obtain

$$a = \frac{-\frac{\kappa^2}{2}}{\frac{v^2}{2} - \frac{\kappa^2}{r}} = \frac{r/2}{1 - \frac{rv^2}{2\kappa^2}} = \frac{r/2}{1-(v/v_{\mathrm{esc}})^2}. \tag{9}$$

The semi-major axis does not depend on the launch angle. When the escape velocity is reached, the orbit's extent becomes infinite.
We also know that the eccentricity is related to the angular momentum constant c (and the energy constant):

$$\kappa^4(e^2-1) = hc^2 \quad \text{(see extra info below)}, \tag{10}$$

which we can solve for e and obtain

$$e = \sqrt{1 + \frac{hc^2}{\kappa^4}}. \tag{11}$$

From $\vec{c} = \vec{r} \times \dot{\vec{r}}$ we can arrive at an equation for c (see lecture),

$$c = r^2\dot{\theta}, \tag{12}$$

which illustrates that the angular momentum depends only on the tangential velocity component, $r\dot{\theta}$. Assuming a launch angle α relative to the surface, we find

$$r\dot{\theta} = v\cos\alpha \qquad \text{and} \qquad c = rv\cos\alpha, \tag{13}$$

and hence,

$$e = \sqrt{1 + \frac{hr^2v^2\cos^2\alpha}{\kappa^4}} = \sqrt{1 - 2\underbrace{\frac{r}{2\kappa^2}}_{1/v_{\mathrm{esc}}^2}\frac{r}{a}v^2\cos^2\alpha}, \tag{14}$$

where we have already substituted $h=-\kappa^2/a$ and can further substitute $v_{\rm esc}$ and our result from above for a,

$$\frac{r}{a}=2\left[1-\frac{v^2}{v_{\rm esc}^2}\right], \tag{15}$$

to obtain

$$e=\sqrt{1-4\left[1-\frac{v^2}{v_{\rm esc}^2}\right]\frac{v^2}{v_{\rm esc}^2}\cos^2\alpha}. \tag{16}$$

For $v\to v_{\rm esc}$ the eccentricity behaves as $e\to 1$. For $v>v_{\rm esc}$, we find $e>1$ and $a<0$, which corresponds to a normal hyperbola. When we launch horizontally, $\alpha=0$, and at the right velocity, $v=v_{\rm esc}/\sqrt{2}$, the stone is injected into a circular orbit ($e=0$).

Bonus: The radius at which the stone touches the surface is given by the equation for an ellipse

$$r\overset{!}{=}\frac{p}{1+e\cos\theta}, \tag{17}$$

which can have up to two solutions for θ, $\theta_\uparrow>0$ for the point where we throw the stone, $\theta_\downarrow<2\pi$ for the point where it falls back down:

$$\theta_{\uparrow\downarrow}=\arccos\left[\frac{1}{e}\left(\frac{p}{r}-1\right)\right]. \tag{18}$$

The stone is above the surface for $\theta_\uparrow<\theta<\theta_\downarrow$. Given that $\theta_\downarrow=2\pi-\theta_\uparrow$, the angular difference between the two solutions is

$$\Delta\theta=2(\pi-\theta_\uparrow)=2\left\{\pi-\arccos\left[\frac{1}{e}\left(\frac{p}{r}-1\right)\right]\right\}=2\arccos\left[\frac{1}{e}\left(1-\frac{p}{r}\right)\right]. \tag{19}$$

The distance along the surface between the two points is then

$$x=2r\arccos\left[\frac{1}{e}\left(1-\frac{p}{r}\right)\right]. \tag{20}$$

Extra info: equation (10) can be derived in the following way. Crossing the angular momentum vector and the Laplace vector, we begin with

$$\vec{c}\times\vec{e}=(\vec{r}\times\dot{\vec{r}})\times\left(\frac{\dot{\vec{r}}\times\vec{c}}{\kappa^2}-\frac{\vec{r}}{r}\right)=\underbrace{\frac{(\vec{r}\times\dot{\vec{r}})\times(\dot{\vec{r}}\times\vec{c})}{\kappa^2}}_{\equiv\vec{A}}-\underbrace{\frac{(\vec{r}\times\dot{\vec{r}})\times\vec{r}}{r}}_{\equiv\vec{B}}. \tag{21}$$

We can now use the relation $\vec{a}\times(\vec{b}\times\vec{d})=\vec{b}(\vec{a}\cdot\vec{d})-\vec{d}(\vec{a}\cdot\vec{b})$ *with* $\vec{a}=(\vec{r}\times\dot{\vec{r}})$, $\vec{b}=\dot{\vec{r}}$, *and* $\vec{d}=\vec{c}$ *to obtain*

$$\begin{aligned}\vec{A}&=\kappa^{-2}\left\{(\vec{r}\times\dot{\vec{r}})\times(\dot{\vec{r}}\times\vec{c})\right\}\\&=\kappa^{-2}\dot{\vec{r}}[\underbrace{(\vec{r}\times\dot{\vec{r}})}_{\vec{c}}\cdot\vec{c}]-\kappa^{-2}\vec{c}[\underbrace{(\vec{r}\times\dot{\vec{r}})\cdot\dot{\vec{r}}}_{=0}]\\&=\kappa^{-2}c^2\dot{\vec{r}},\end{aligned} \tag{22}$$

and for $\vec{a}=\vec{b}=\vec{r}$ *and* $\vec{d}=\dot{\vec{r}}$,

$$\begin{aligned}\vec{B}&=r^{-1}(\vec{r}\times\dot{\vec{r}})\times\vec{r}\\&=-r^{-1}\vec{r}(\underbrace{\vec{r}\cdot\dot{\vec{r}}}_{r\dot{r}})+r^{-1}\dot{\vec{r}}(\underbrace{\vec{r}\cdot\vec{r}}_{r^2})\\&=-\dot{r}\vec{r}+r\dot{\vec{r}}.\end{aligned} \tag{23}$$

Using $\vec{a}\cdot(\vec{b}\times\vec{d}) = \vec{b}\cdot(\vec{d}\times\vec{a})$ with $\vec{a}=\vec{c}\times\vec{e}$, $\vec{b}=\vec{c}$, and $\vec{d}=\vec{e}$, we proceed to consider

$$
\begin{aligned}
(\vec{c}\times\vec{e})\cdot(\vec{c}\times\vec{e}) &= \left[\kappa^{-2}c^2\dot{\vec{r}} + \dot{r}\vec{r} - r\dot{\vec{r}}\right]\cdot\left[\kappa^{-2}c^2\dot{\vec{r}} + \dot{r}\vec{r} - r\dot{\vec{r}}\right] \\
\vec{c}\cdot[\vec{e}\times(\vec{c}\times\vec{e})] &= \left[\left(\kappa^{-2}c^2 - r\right)\dot{\vec{r}} + \dot{r}\vec{r}\right]\cdot\left[\left(\kappa^{-2}c^2 - r\right)\dot{\vec{r}} + \dot{r}\vec{r}\right] \\
\vec{c}\cdot[\vec{c}(\vec{e}\cdot\vec{e}) - \vec{e}(\vec{e}\cdot\vec{c})] &= \left(\kappa^{-2}c^2 - r\right)^2 v^2 + \dot{r}^2 r^2 + 2\left(\kappa^{-2}c^2 - r\right)\dot{r}^2 r \\
(\underbrace{\vec{c}\cdot\vec{c}}_{=c^2})(\underbrace{\vec{e}\cdot\vec{e}}_{=e^2}) - (\underbrace{\vec{c}\cdot\vec{e}}_{=0})(\underbrace{\vec{e}\cdot\vec{c}}_{=0}) &= \left(\kappa^{-4}c^4 - 2\kappa^{-2}c^2 r + r^2\right)v^2 + \left(2\kappa^{-2}c^2 r - r^2\right)\dot{r}^2.
\end{aligned}
\tag{24}
$$

The left-hand side could be understood more easily, when considering that $\vec{c}\perp\vec{e}$ implies $|\vec{c}\times\vec{e}| = |\vec{c}||\vec{e}| = ce$. For the right-hand side, the angular momentum constant, $c = r^2\dot{\theta}$, can now be used to express the radial velocity in terms of orbital velocity,

$$\dot{r}^2 = v^2 - r^2\dot{\theta}^2 = v^2 - c^2 r^{-2}, \tag{25}$$

which allows to simplify further

$$
\begin{aligned}
c^2e^2 &= \left(\kappa^{-4}c^4 - 2\kappa^{-2}c^2 r + r^2\right)v^2 + \left(2\kappa^{-2}c^2 r - r^2\right)\left(v^2 - c^2 r^{-2}\right) \\
c^2e^2 &= \kappa^{-4}c^4 v^2 - 2\kappa^{-2}c^4 r^{-1} + c^2 \\
c^2(e^2-1) &= \kappa^{-4}c^4(\underbrace{v^2 - 2\kappa^2 r^{-1}}_{\equiv h}) \\
\kappa^4(e^2-1) &= hc^2,
\end{aligned}
\tag{26}
$$

where h is the usual energy constant.