Celestial Mechanics – Solutions

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Unit 2

Problem 2.1

The barycenter of the solar system is close to the common center of mass of the Sun and the most massive planet, which is Jupiter. Let $M_\odot$ be the solar mass, $R_\odot$ its radius, M_J the Jupiter mass, and r_J its heliocentric distance (the radius of its orbit). Further, denote by x the distance of the barycenter from the center of the Sun.
Then,

$$M_\odot x = M_\mathrm{J}(r_\mathrm{J} - x)$$

or

$$M_\odot x \approx M_\mathrm{J} r_\mathrm{J},$$

which we can re-write as

$$\frac{x}{R_\odot} \approx \frac{M_\mathrm{J}}{M_\odot}\frac{r_\mathrm{J}}{R_\odot}.$$

This shows that the barycenter's distance from the solar center, measured in the solar radii, is just the mass ratio of Jupiter and the Sun times the ratio of the orbital radius of Jupiter and the solar radius. Coincidentally, the product of the two ratios is close to unity. Indeed, $M_\mathrm{J}/M_\odot \approx 1/1000$, whereas $r_\mathrm{J} \approx 5\,\mathrm{au} \approx 1000 R_\odot$. Thus $x/R_\odot \approx 1$, and the barycenter is *close to* the solar surface.

A more accurate calculation gives

$$\frac{x}{R_\odot} \approx \frac{M_\mathrm{J}}{M_\odot + M_\mathrm{J}} \cdot \frac{r_\mathrm{J}}{R_\odot} \approx 9.53 \times 10^{-4}\, \frac{5.203 \times (1.496 \times 10^{11}\,\mathrm{m})}{6.96 \times 10^{8}\,\mathrm{m}} \approx 1.065$$

suggesting that the barycenter is slightly *outside* the solar photosphere.

However, even this calculation still may not be accurate enough. For instance, the Jupiter orbit is slightly elliptic, making r_J vary. A yet more significant effect is the effect of the other planets. For instance, Saturn can induce a correction to $x/R_\odot$ of the order $(M_\mathrm{S}/M_\odot)(r_\mathrm{S}/R_\odot) \approx (1/3000)(2000/1) \sim 0.6$. Therefore, if Jupiter and Saturn are on opposite sides of the Sun, the barycenter of the solar system will be *within* the Sun ...

Problem 2.2

Assuming a body to be spherical, from the energy integral we obtain

$$v_\mathrm{esc} = \sqrt{2\frac{GM}{R}} = \sqrt{2\frac{4\pi G\rho R^3/3}{R}} = R\sqrt{\frac{8\pi G\rho}{3}} \propto R\sqrt{\rho}.$$

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Table 1: Masses and semimajor axes of the Solar System planets

	$m/M_\odot$	$a/R_\odot$	$x/R_\odot$
Mercury	1.66×10^{-7}	83	1.4×10^{-5}
Venus	2.45×10^{-6}	155	3.8×10^{-4}
Earth	3.00×10^{-6}	215	6.5×10^{-4}
Mars	3.22×10^{-7}	328	1.1×10^{-4}
Jupiter	9.55×10^{-4}	1119	1.07
Saturn	2.86×10^{-4}	2061	0.58
Uranus	4.37×10^{-5}	4130	0.18
Neptune	5.15×10^{-5}	6547	0.33

The special case with $\rho = 1$ g cm^{-3} and $R = 1$ km results in $v_{\rm esc} = 0.75$ m/s, which can be combined with the above to

$$\frac{v_{\rm esc}}{0.75\ \text{m/s}} = \frac{R}{1\ \text{km}}\sqrt{\frac{\rho}{1\ \text{g cm}^{-3}}}.$$

or

$$v_{\rm esc} = 0.75\ \text{m/s}\ \times\ \frac{R}{1\ \text{km}}\sqrt{\frac{\rho}{1\ \text{g cm}^{-3}}}.$$

The density ρ is ranging from 0.7 g cm^{-3} for Saturn up to 5.5 g cm^{-3} for Earth, while the typical densities for planetesimals are between 1 and 3 g cm^{-3}. Since the dependence on ρ is weak ($\propto \sqrt{\rho}$) and the values are all within the same order of magnitude, we find

$$v_{\rm esc} \approx 1\ \text{m/s}\ \times\ \frac{R}{\text{km}}.$$

For Earth with $R \approx 6000$ km, we predict $v_{\rm esc} \approx 6$ km/s (actual value: 11 km/s). For Phobos with $R \approx 12$ km, we predict $v_{\rm esc} \approx 12$ m/s (actual value: close to it, slightly depends on the ejection point, as Phobos has a potato form).

Problem 2.3

In that strange universe, the gravitational force exerted by mass 1 on mass 2 scales as

$$\vec{F} = -\frac{GM_1M_2}{r^4}\vec{r} \tag{1}$$

where the value and units of G are different from our normal gravitational constant.

Equation of Motion

Relative to the origin (see Fig. 1), we have

$$\ddot{\vec{\rho}}_2 = \frac{\vec{F}}{M_2} = -\frac{GM_1}{r^4}\vec{r},$$

and

$$\ddot{\vec{\rho}}_1 = -\frac{\vec{F}}{M_1} = \frac{GM_2}{r^4}\vec{r}.$$

From the vectors in Fig. 1, we find

$$\vec{r} = \vec{\rho}_2 - \vec{\rho}_1,$$

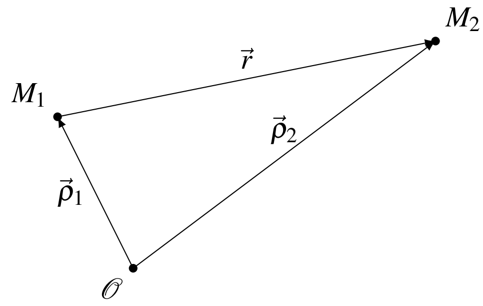

Figure 1: The positions of two masses and the coordinate origin $\mathscr{O}$.

and so

$$\ddot{\vec{r}} = \ddot{\vec{\rho}}_2 - \ddot{\vec{\rho}}_1.$$

Substituting $\ddot{\vec{\rho}}_1$ and $\ddot{\vec{\rho}}_2$ from above, we arrive at the equation of motion (EoM):

$$\ddot{\vec{r}} + \frac{G(M_1 + M_2)}{r^4}\vec{r} = \ddot{\vec{r}} + \frac{\kappa^2}{r^4}\vec{r}0,$$

where we defined $\kappa^2 \equiv G(M_1 + M_2)$.

Angular Momentum Integral

Crossing $\vec{r}$ with EoM:

$$\vec{r} \times \ddot{\vec{r}} + \frac{\kappa^2}{r^4}\vec{r} \times \vec{r} = 0.$$

Now $\vec{r} \times \vec{r} = 0$, so from above

$$\vec{r} \times \ddot{\vec{r}} = 0.$$

Integrate this with respect to time:

$$\vec{I} = \int \vec{r} \times \ddot{\vec{r}}\,\mathrm{d}t.$$

We can solve this by guessing the solution and checking its validity. From the chain rule we know

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \ddot{\vec{r}} + \dot{\vec{r}} \times \dot{\vec{r}} = \vec{r} \times \ddot{\vec{r}}, \tag{2}$$

where again we used $\dot{\vec{r}} \times \dot{\vec{r}} = 0$. So the solution to our integral is indeed

$$\vec{I} = \int \vec{r} \times \ddot{\vec{r}}\,\mathrm{d}t = \vec{r} \times \dot{\vec{r}} - \vec{c},$$

where $\vec{c}$ is an arbitrary constant. Hence we arrive at the angular momentum integral:

$$\vec{r} \times \dot{\vec{r}} = \vec{c}.$$

Angular momentum as we know it is still in this other universe.

Energy Integral

The scalar product of EoM with $\dot{\vec{r}}$ yields

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{\kappa^2}{r^4} \dot{\vec{r}} \cdot \vec{r} = 0. \tag{3}$$

Now note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) = \dot{\vec{r}} \cdot \frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t} + \dot{\vec{r}} \cdot \frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t} = 2\dot{\vec{r}} \cdot \frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t} = 2\dot{\vec{r}} \cdot \ddot{\vec{r}},$$

so we can change $\dot{\vec{r}} \cdot \ddot{\vec{r}}$ to

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) = \frac{1}{2}\frac{\mathrm{d}\dot{\vec{r}}^2}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t}. \tag{4}$$

Noting that $\dot{r}$ is simply the projection of $\dot{\vec{r}}$ onto $\dot{r}$, we can use

$$\vec{r} \cdot \dot{\vec{r}} = r\dot{r}. \tag{5}$$

Applying the substitutions in eqs. (4) and (5) to eq. (3), we obtain

$$\frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t} + \kappa^2 \frac{\dot{r}}{r^3} = 0. \tag{6}$$

The $\dot{r}/r^3$ in the second term can then be rewritten as

$$\frac{\dot{r}}{r^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2r^2}\right) \tag{7}$$

and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right) - \kappa^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2r^2}\right) = 0, \tag{8}$$

which can be integrated to

$$\frac{v^2}{2} - \frac{\kappa^2}{2r^2} = \frac{h}{2}. \tag{9}$$

There is again an energy constant $h/2$, albeit with a slightly different contribution from the potential term.

Circular Orbits

With the angular momentum integral being the same in this other universe, we can relate radial and angular variables in the same way as in ours:

$$c = r^2\dot{\theta} \quad \text{or} \quad \mathrm{d}t = \frac{r^2}{c}\mathrm{d}\theta. \tag{10}$$

The azimuthal velocity component can be written as

$$r\dot{\theta} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{c}{r}, \tag{11}$$

and hence, the orbital velocity as

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{c^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{c^2}{r^2} \tag{12}$$

The energy integral ($\times 2$) then reads

$$h = v^2 - \frac{\kappa^2}{r^2} = \frac{c^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{c^2}{r^2} - \frac{\kappa^2}{r^2}. \tag{13}$$

Substituting Binet's variable, $u \equiv 1/r$ and $\mathrm{d}r/\mathrm{d}\theta = \mathrm{d}(1/u)/\mathrm{d}\theta = -u'/u^2$, we find

$$h = u'^2 c^2 + u^2 c^2 - u^2 \kappa^2. \tag{14}$$

Another differentiation by θ leads to

$$0 = 2u''u'c^2 + 2u'u(c^2 - \kappa^2) \tag{15}$$

$$u'' = u \underbrace{\left(\frac{\kappa^2}{c^2} - 1\right)}_{\equiv A}. \tag{16}$$

Depending on the sign of A, we find three types of real-valued solutions:

$$u(\theta) = \begin{cases} u_0 \exp\left[\theta\sqrt{A}\right] + u_1 \exp\left[-\theta\sqrt{A}\right] & \text{for } A > 0 \quad (c < \kappa), \\ u_0 + u_1\theta & \text{for } A = 0 \quad (c = \kappa), \\ u_0 \cos\left[\theta\sqrt{-A} + \text{const}\right] & \text{for } A < 0 \quad (c > \kappa), \end{cases} \tag{17}$$

where u_0 and u_1 are constants that depend on the initial conditions $u(0)$ and $u'(0)$. The exponential solution has u grow and r reduce for $\theta \to \infty$, corresponding to an inward spiral. The cosine solution has an upper boundary that corresponds to an minimum distance $r_{\min}$, but $u(\theta)$ can reach zero (and below) corresponding to $\to \infty$, i. e. an unbound orbit. For $A = 0$ (corresponding to $\kappa = c$) we have another spiral, open inward or outward, depending on $u_1 = u'(0)$. For the special case of $u'(0) = 0$, the solution describes a circle. This solution is, however, not stable against small perturbations.

Alternatively, we can check which type of radial motion is possible in the effective potentials $U(r)$ in both universes:

$$\frac{\dot{r}^2}{2} + U(r) = \text{const}. \tag{18}$$

For the normal universe, we obtain

$$\frac{\dot{r}^2}{2} + \underbrace{\frac{c^2}{2r^2} - \frac{\kappa^2}{r}}_{\equiv U_{\text{normal}}} = \frac{h}{2}. \tag{19}$$

Here, U_{normal} can have a local minimum, corresponding to motion between a maximum and a minimum radius. For the strange universe, we find

$$\frac{\dot{r}^2}{2} + \underbrace{\frac{c^2 - \kappa^2}{2r^2}}_{\equiv U_{\text{strange}}} = \frac{h}{2}, \tag{20}$$

which does not have a local extremum, and hence, only motion that is open outward or open inward, depending on the sign of U. Circular motion is only possible when $U_{\text{strange}} = h = 0 = \text{const}$, which is the case for $|c| = \kappa$. Even then, the flat potential for that case means that the orbit would be prone to radial drift.