Celestial Mechanics – Solutions

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Unit 1

Problem 1.1

It is clear that the surface is axially symmetric with respect to the x axis. Therefore, it is sufficient to find the section of the surface by the x–y plane, i. e. the equation of the form $f(x,y) = 0$ (see Fig. 1).
By writing down the equality of the gravitational forces from the Earth and the Sun:

$$M_\odot / r_\odot^2 = M_\oplus / r_\oplus^2,$$

and taking into account that

$$r_\oplus^2 = x^2 + y^2, \qquad r_\odot^2 = (x-a)^2 + y^2,$$

we can perform the following series of transformations:

$$\begin{aligned}
M_\odot / r_\odot^2 &= M_\oplus / r_\oplus^2 \\
M_\odot r_\oplus^2 &= M_\oplus r_\odot^2 \\
M_\odot \left(x^2 + y^2\right) &= M_\oplus \left[(x-a)^2 + y^2\right] \\
x^2(M_\odot - M_\oplus) + 2xaM_\oplus + y^2(M_\odot - M_\oplus) &= a^2 M_\oplus \\
x^2 + 2xa\frac{M_\oplus}{M_\odot - M_\oplus} + y^2 &= a^2 \frac{M_\oplus}{M_\odot - M_\oplus} \\
\left[x + a\frac{M_\oplus}{M_\odot - M_\oplus}\right]^2 - \left[a\frac{M_\oplus}{M_\odot - M_\oplus}\right]^2 + y^2 &= a^2 \frac{M_\oplus}{M_\odot - M_\oplus} \\
\left[x + a\frac{M_\oplus}{M_\odot - M_\oplus}\right]^2 + y^2 &= a^2 \frac{M_\oplus}{M_\odot - M_\oplus}\left[\frac{M_\oplus}{M_\odot - M_\oplus} + 1\right] \\
\left[x + \underbrace{a\frac{M_\oplus}{M_\odot - M_\oplus}}_{\equiv \Delta x}\right]^2 + y^2 &= \underbrace{a^2 \frac{M_\oplus}{M_\odot - M_\oplus}\left[\frac{M_\odot}{M_\odot - M_\oplus}\right]}_{\equiv R^2}.
\end{aligned}$$

The resulting equation does, indeed, describe a sphere. Its radius is

$$R = a\sqrt{\frac{M_\odot M_\oplus}{(M_\odot - M_\oplus)^2}} \approx a\sqrt{\frac{M_\oplus}{M_\odot}}.$$

Its center is shifted from the center of the Earth along the x axis away from the Sun by the distance

$$\Delta x = a\frac{M_\oplus}{M_\odot - M_\oplus} \approx a\frac{M_\oplus}{M_\odot}.$$

Numerically, $R = 260\,000\,\mathrm{km}$. The orbit of the Moon is substantially farther, so the Sun attracts the Moon stronger than the Earth does – a well-known paradox. Also, $\Delta x = 450\,\mathrm{km}$, so the center of the sphere is inside the Earth.
The gravitation sphere would be a plane (that includes the midpoint between the Earth and the Sun and is perpendicular to the line that connects the two) if the mass of the Earth were exactly equal to the mass of the Sun. In that case, the center of the sphere would be at (negative) infinity.

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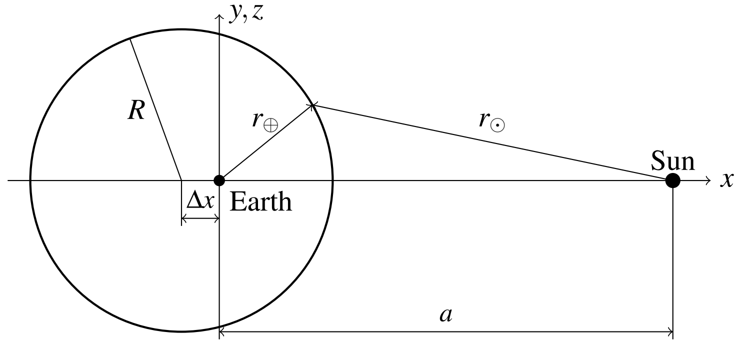

Figure 1: Sphere on which the gravitational forces towards Earth and towards the Sun are of the same magnitude.

Problem 1.2

We choose a cylindric coordinate system centred on the disc mid-point. Then, each point in the disc (of radius R and total mass M) is defined through its distance from the centre r' and an azimuthal angle φ. Its distance to the test particle (of mass m_T) is $x^2 + r'^2$, and the integral to be solved is given by

$$F_\mathrm{x} = -G m_\mathrm{T} \Sigma \int_0^R \int_0^{2\pi} \frac{x}{r'^2 + x^2} \frac{1}{(r'^2 + x^2)^{1/2}} \, r' \mathrm{d}\varphi \, \mathrm{d}r',$$

where $\Sigma \equiv M/(\pi R^2)$ is the disc's surface mass density (which we assume constant). After simplification and trivial integration over φ we have

$$F_\mathrm{x} = -2\pi G m_\mathrm{T} \Sigma \int_0^R \frac{x}{(r'^2 + x^2)^{3/2}} \, r' \mathrm{d}r'.$$

Integration over r' results in

$$F_\mathrm{x}(x) = 2\pi G m_\mathrm{T} \Sigma \left[\frac{x}{(R^2 + x^2)^{1/2}} - \frac{x}{(x^2)^{1/2}} \right] = 2\pi G m_\mathrm{T} \Sigma \left[\frac{x}{(R^2 + x^2)^{1/2}} - \frac{x}{|x|} \right]. \tag{1}$$

Due to the disc being infinitely thin, the $x/|x|$ makes F_x jump from -1 to 1 at $x = 0$. In order to get the asymptote for $|x| \ll R$, we take a look at the Taylor-series expansion of $F(x)$ which is

$$F_\mathrm{x}(x) = 2\pi G m_\mathrm{T} \Sigma \left[\frac{x}{R} \mp 1 + \mathcal{O}(x^2) \right], \tag{2}$$

where the upper sign is for $x > 0$ and the lower one for $x < 0$. The value of $F_\mathrm{x}(x)$ at the surface of the disc, i.e. at $x/R \approx 0$, is independent of the disc's extent and equal to that of an infinite plane.
Both the full solution (Eq. 1) and the linear approximation (Eq. 2) are plotted in Fig. 2.

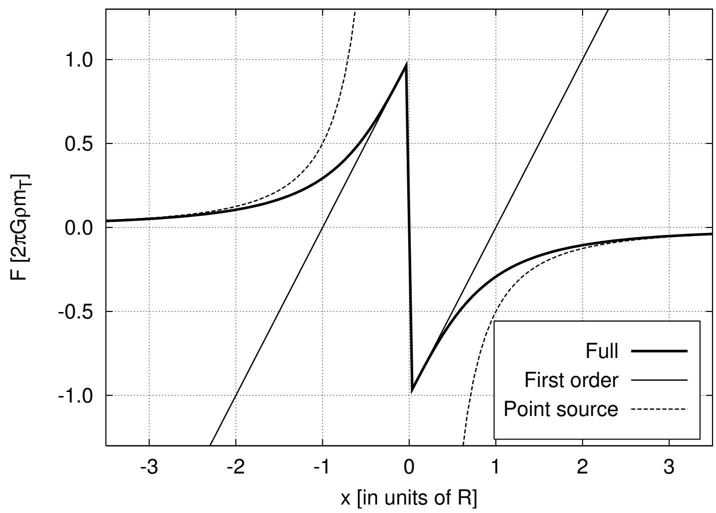

Figure 2: The function $F_{\mathrm{x}}(x)$ (for $R = 2$), given in Eq. (1), its asymptote for $|x| \ll R$, given in Eq. (2), and its asymptote for $|x| \gg R$, where the disc becomes a point source of mass $\pi R^2 \Sigma$.