Celestial Mechanics – Exercises

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Unit 9

Problem 9.1

Consider a Hamiltonian system

$$\frac{\mathrm{d}q_k}{\mathrm{d}t} = \frac{\partial \mathscr{H}}{\partial p_k}, \qquad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{\partial \mathscr{H}}{\partial q_k}$$

with variables (p_k, q_k). A transformation of variables from (p_k, q_k) to (P_k, Q_k) is called *canonical* if the equations in the new variables also have a Hamiltonian form. A particular case of canonical transformations is *contact transformations*, which preserve not only the Hamiltonian form, but also the Hamiltonian itself. For a transformation to be a contact transformation, it is sufficient that

$$\sum_k (p_k \, \mathrm{d}q_k - P_k \, \mathrm{d}Q_k) = 0. \tag{1}$$

Starting with the Delaunay-Hamiltonian

$$\mathscr{H} = \frac{\kappa^4}{2L_1^2} + R,$$

show that the transformation from the Delaunay variables (l_k, L_k) to the new set of variables $(\tilde{l}_k, \tilde{L}_k)$, with "coordinates" $\tilde{L}_k$ given by

$$\tilde{L}_1 = L_1, \qquad \tilde{L}_2 = L_1 - L_2, \qquad \tilde{L}_3 = L_2 - L_3, \tag{2}$$

is a contact transformation. Find the "momenta" $\tilde{l}_k$. Express $(\tilde{l}_k, \tilde{L}_k)$ through usual Keplerian elements.
(**3 points**)

Imagine you want to study the motion of planets in the solar system, using the perturbation equations in the Hamiltonian form. Why are the Poincaré variables better than the Delaunay ones?
(**+1 point**)

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