Physics of Planetary Systems — Exercises
Suggested Solutions to Set 9

Problem 9.1

(2 points)

Querying the catalog for HIP 66074 ($\mathcal{M}_* \approx 0.7\ \mathcal{M}_\odot$) results in an astrometric solution with a parallax

$$\varpi = (28.211 \pm 0.012)\ \text{mas}, \tag{1}$$

(corresponding to a distance $d = 1\ \text{au}/\varpi = 35.5$ pc), an apparent semi-major axis

$$\alpha = (0.213 \pm 0.033)\ \text{mas}, \tag{2}$$

and an orbital period

$$P = (297.6 \pm 2.7)\ \text{d}. \tag{3}$$

The orbital period allows us access to the sum of the masses, $\mathcal{M} \equiv \mathcal{M}_* + \mathcal{M}_\text{p}$, and semi-major axes, $a = a_* + a_\text{p}$:

$$P = 2\pi\sqrt{\frac{a^3}{G\mathcal{M}}}, \tag{4}$$

where

$$a_*\mathcal{M}_* = a_\text{p}\mathcal{M}_\text{p} = (a - a_*)\mathcal{M}_\text{p} \tag{5}$$

and hence,

$$\mathcal{M} = \mathcal{M}_* \frac{a}{a - a_*}, \tag{6}$$

resulting in

$$P = 2\pi\sqrt{\frac{a^2(a-a_*)}{G\mathcal{M}_*}} \qquad \text{or} \qquad a^3 - a^2 a_* - \underbrace{\frac{G\mathcal{M}_* P^2}{4\pi^2}}_{\equiv\xi} = 0. \tag{7}$$

This cubic equation can be solved analytically (see extra info below), but it becomes much simpler if we assume $\mathcal{M}_\text{p} \ll \mathcal{M}_*$, and hence, $a \approx a_\text{p} \gg a_*$:

$$a_\text{p} \approx \sqrt[3]{\frac{G\mathcal{M}_* P^2}{4\pi^2}} = 0.77\ \text{au}. \tag{8}$$

The stellar semi-major axis is

$$a_* = \alpha d = \alpha \frac{1\ \text{au}}{\varpi} = 0.0075\ \text{au}, \tag{9}$$

from which we obtain the planet mass

$$\mathcal{M}_\text{p} = \frac{a_*}{a_\text{p}}\mathcal{M}_* = 0.0097\,\mathcal{M}_* \approx 7.2\,\mathcal{M}_\text{Jup} \ll \mathcal{M}_*. \tag{10}$$

Extra info: the full solution to the cubic eq. (7) is

$$a = \frac{a_*}{3}\left[1 + \frac{1}{C} + C\right], \tag{11}$$

where the middle term dominates for $\mathcal{M}_* \gg \mathcal{M}_\text{p}$ because

$$C = \sqrt[3]{\frac{27\xi}{2a_*^3} + 1 - \sqrt{\left(\frac{27\xi}{2a_*^3}\right)^2 + 27\frac{\xi}{a_*^3}}} = \sqrt[3]{\frac{27\xi}{4a_*^3}\left(\sqrt{1 + \frac{4a_*^3}{27\xi}} - 1\right)^2} = \frac{a_*}{3\sqrt[3]{\xi}}\left[1 - \frac{2a_*^3}{81\xi} + \mathcal{O}(a_*^6)\right] \ll 1 \tag{12}$$

and $\sqrt[3]{\xi} \approx a_\text{p} \gg a_*$.

Problem 9.2 *(2 points)*

If you have a telescope with focal length $F = 3 \times 39$ m $= 117$ m, the plate scale will be

$$\text{plate scale} = \frac{360 \times 60 \times 60''}{2\pi F} \approx \frac{206\,265''}{F} = \frac{206\,265''}{117\,000 \text{ mm}} = 1.76''/\text{mm}. \tag{13}$$

For a pixel size of 15 µm, this corresponds to a plate scale of 0.02644″/pixel. A displacement by one hundreth of a pixel corresponds to an astrometric perturbation of 0.0002644″ (0.2644 mas).
The astrometric perturbation is

$$\theta[''] = \frac{\mathscr{M}_{\text{planet}}[\mathscr{M}_\odot]}{\mathscr{M}_*[\mathscr{M}_\odot]} \frac{a[\text{au}]}{d[\text{pc}]}, \tag{14}$$

where a is the semi-major axis and d is the distance to α Cen. This gives the planet mass

$$\mathscr{M}_{\text{planet}}[\mathscr{M}_\odot] = \theta['']\frac{\mathscr{M}_*[\mathscr{M}_\odot] d[\text{pc}]}{a[\text{au}]} = 2.64 \times 10^{-4} \frac{1 \times 1.34}{1} = 3.53 \times 10^{-4} \quad \Rightarrow \quad \mathscr{M}_{\text{planet}} = 0.37\ \mathscr{M}_{\text{Jupiter}}. \tag{15}$$

Problem 9.3 *(2 points)*

The 3D equation of mass growth rate is:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \approx \rho \sigma v_{\text{rel}} \tag{16}$$

In 2D, the following changes are needed. Firstly, ρ is replaced by surface density, Σ. Secondly, the cross section for collision σ without gravitational enhancement is $2s$ instead of πs^2. With gravitational enhancement, it is therefore

$$\sigma = 2s \left(1 + \frac{v_{\text{esc}}^2}{v_{\text{rel}}^2}\right)^{1/2} \tag{17}$$

instead of

$$\sigma = \pi s^2 \left(1 + \frac{v_{\text{esc}}^2}{v_{\text{rel}}^2}\right). \tag{18}$$

Collecting all results together and using $v_{\text{rel}} \ll v_{\text{esc}}$ as in the 3D case, we obtain the 2D equation of mass growth rate:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \approx \Sigma \cdot 2s \cdot v_{\text{esc}} \tag{19}$$

or

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \propto \Sigma \mathscr{M}^{2/3} \tag{20}$$

where the power of the mass in the right-hand side is less than unity, so it is not a runaway growth. The exponent (2/3) is the same as for the oligarchic growth.

Problem 9.4 *(2 points)*

The mass of finished oligarchs is given by

$$\mathscr{M}_{\text{iso}} = \frac{(2\pi b \Sigma)^{3/2} r^3}{(3\mathscr{M}_*)^{1/2}} \tag{21}$$

Assume $b = 10$ and $\Sigma = 10$ g cm^{-2} at 1 au. Then

$$\begin{aligned}\mathscr{M}_{\rm iso} &= \frac{(2\cdot 3\cdot 10\cdot 10)^{3/2}(1.5\cdot 10^{13})^3}{(3\cdot 2\cdot 10^{33})^{1/2}} \approx \frac{600^{3/2}\cdot 3\cdot 10^{39}}{(60\cdot 10^{32})^{1/2}} \approx \frac{600\cdot 25\cdot 3\cdot 10^{39}}{8\cdot 10^{16}} \approx \frac{600\cdot 10\cdot 10^{39}}{10^{16}} \\ &\approx 6\cdot 10^{26}\ \text{g} \approx 0.1\mathscr{M}_\oplus\end{aligned}$$

To get the result at 5 au, we have to multiply this by $(3/10)^{3/2}$ (to account for difference in Σ) and by $(5/1)^3$ (to account for difference in r). This gives:

$$\mathscr{M}_{\rm iso} \approx 0.1\mathscr{M}_\oplus \cdot (3/10)^{3/2}\cdot 5^3 \approx 0.1\mathscr{M}_\oplus \cdot 20 \approx 2\mathscr{M}_\oplus. \tag{22}$$

The orbital separation of isolated oligarchs is given by

$$\Delta r = br_{\rm H} = br\left(\frac{\mathscr{M}_{\rm iso}}{3M_*}\right)^{1/3} \tag{23}$$

Numerically, at 1au

$$\Delta r \approx 10\cdot 1.5\cdot 10^{13}\left(\frac{6\cdot 10^{26}}{3\cdot 2\cdot 10^{33}}\right)^{1/3} \approx 1.5\cdot 10^{14}\left(\frac{1}{10\cdot 10^6}\right)^{1/3} \approx 10^{12}\ \text{cm} \approx 0.07\text{au} \tag{24}$$

Again, to get the result at 5 au, we have to multiply this by 5 (to account for the difference in r) and with $20^{1/3}$ (to account for difference in $\mathscr{M}_{\rm iso}$). This gives:

$$\Delta r \approx 0.07\ \text{au}\cdot 5\cdot 2.7 \approx 1\ \text{au} \tag{25}$$