Physics of Planetary Systems — Exercises
Suggested Solutions to Set 7

Problem 7.1 *(2 points)*

To be able to integrate the growth, it is easier to first convert from masses to radii. The relation

$$m = \frac{4\pi\rho s^3}{3} \tag{1}$$

corresponds to

$$\dot{m} = 4\pi\rho s^2 \dot{s}. \tag{2}$$

This can be combined with the given growth rate,

$$\dot{m} = -\text{const} \times s^2 \rho_{\text{gas}} \dot{z}, \tag{3}$$

to obtain

$$4\pi\rho s^2 \dot{s} = -\text{const} \times s^2 \rho_{\text{gas}} \dot{z}, \tag{4}$$

and hence,

$$4\pi\rho \mathrm{d}s = -\text{const} \times \rho_{\text{gas}} \mathrm{d}z. \tag{5}$$

Note that the constant is negative because the altitude z decreases. Both sides of the equation can be integrated separately, resulting in

$$4\pi\rho \left(s_{\text{final}} - s_0\right) = -\text{const} \times \int_{z_0}^{0} \rho_{\text{gas}} \mathrm{d}z \approx \text{const} \times \frac{\Sigma_{\text{gas}}}{2}, \tag{6}$$

because the column density (or surface mass density) is related to the volume mass density via

$$\Sigma_{\text{gas}} = \int_{-\infty}^{\infty} \rho_{\text{gas}} \mathrm{d}z, \tag{7}$$

and $z_0 \approx \infty$. Solving for the final grain radius, we obtain

$$s_{\text{final}} = s_0 + \text{const} \times \frac{\Sigma_{\text{gas}}}{8\pi\rho}. \tag{8}$$

As long as the initial radius is small ($0 \approx s_0 \ll s_{\text{final}}$), the final radius is independent from it:

$$s_{\text{final}} \approx \text{const} \times \frac{\Sigma_{\text{gas}}}{8\pi\rho}. \tag{9}$$

Bonus problem 7.2 *(2 extra points)*

Deriving an actual value for s_{final} requires estimates for Σ, ρ, and in particular, "const". For the gas column density and the grain bulk density we can assume typical values (at ~ 1 au):

$$\Sigma_{\text{gas}} \sim 1000 \text{ g/cm}^2, \quad \rho \sim 1 \text{ g/cm}^3. \tag{10}$$

The constant can be obtained from a comparison with the growth equation given in the lecture:

$$\dot{m} = \sigma \rho_{\text{dust}} v_{\text{sett}}, \tag{11}$$

where $v_{\mathrm{sett}} = -\dot{z}$, $\sigma = \pi s^2$, and $\rho_{\mathrm{dust}} \sim 0.01\rho_{\mathrm{gas}}$. Hence we find

$$\dot{m} \sim -0.01\pi s^2 \rho_{\mathrm{gas}} \dot{z}, \tag{12}$$

and comparison with equation (3) shows that

$$\mathrm{const} \sim 0.01\pi. \tag{13}$$

With these numbers, the final radius is then given by

$$s_{\mathrm{final}} \sim 0.01\pi \times \frac{1000 \text{ g/cm}^2}{8\pi \times 1 \text{ g/cm}^3} \sim \frac{10}{8} \text{ cm} \sim 1 \text{ cm}. \tag{14}$$

Problem 7.3 *(1 point)*

On Earth's surface, the vertical free-fall acceleration is given by

$$\ddot{z} = -g \quad \overset{\int \ldots \mathrm{d}t}{\Longrightarrow} \quad \dot{z}(t) - \dot{z}(0) = -gt \quad \overset{\int \ldots \mathrm{d}t}{\Longrightarrow} \quad z(t) - z(0) - t\dot{z}(0) = -\frac{1}{2}gt^2, \tag{15}$$

where z is height, $\dot{z}$ vertical speed, and $g \approx 9.8 \text{ m/s}^2$ the free-fall constant (which we assume to be constant). Hence, we find

$$z(0) = z(t) - t\dot{z}(0) + \frac{1}{2g}\left[\dot{z}(0) - \dot{z}(t)\right]^2, \tag{16}$$

which can be simplified to

$$z(0) = \frac{1}{2g}\left[\dot{z}(t)\right]^2 \tag{17}$$

if we assume start at rest ($\dot{z}(0) = 0$) and finish on the ground ($z(t) = 0$). For $\dot{z}(t) = -1$ cm/s, the required initial altitude is then

$$z(0) \approx \frac{1}{20 \text{ m/s}^2}\left[0.01 \text{ m/s}\right]^2 = 5 \text{ µm}, \tag{18}$$

which is surprisingly small!