Physics of Planetary Systems — Exercises
Suggested Solutions to Set 4

Problem 4.1 *(2 points)*

To calculate this, one needs to know about magnitudes, the fact that the RV error is roughly proportional to $(S/N)^{-1}$ (where S/N is the signal-to-noise ratio), please see the expression for the predicted radial velocity error in the lecture,

$$\sigma[\text{m/s}] = \frac{2.4 \cdot 10^{11}}{R^{3/2} \cdot (S/N) \cdot (\Delta\lambda[\AA])^{1/2}}$$

and that S/N is proportional to the square root of the number of detected photons, because it is dominated by photon noise (shot noise):

$$S/N \propto \frac{S}{\sqrt{S}} \propto \sqrt{S}$$

So we need to calculate the number of detected photons on the fainter star, given by the fainter magnitude, plus the fact that one will expose longer: A magnitude difference of $\Delta m = 6$ corresponds to a brightness (photons) reduction by a factor of $2.512^{\Delta m} = 2.512^6 = 251.2$ or can alternatively be calculated using the formula

$$\Delta m = 2.5 \ \log \frac{F_1}{F_2}.$$

This means in 30 minutes on a 15th magnitude star, the HARPS spectrograph will detect 1/251.2 the number of photons (i. e. $1/\sqrt{251.2} = 1/15.85$ times the signal to noise ratio since $S/N \propto \sqrt{\text{detected photons}}$). But by increasing the exposure time, one detects more photons by a factor $60/30 = 2$. So compared to the $m_\text{v} = 9$ mag with a 30 min exposure, on a $m_\text{v} = 15$ mag star HARPS will detect in one hour $(1/251.2) \times 2 = 1/125.6$ times the number of photons. Since RV error is inversely proportional to S/N and the S/N is proportional to the square root of the detected photons, one will have an RV error 1 m/s $\times \sqrt{125.6} = 11.2$ m/s.
Note: at these faint magnitudes systematic errors (e. g. characteristics of the detector) become an important contribution, so the error may be much larger than for photon statistics.

Problem 4.2 *(2 points)*

Methods to precisely measure stellar RVs mostly differ by the way the wavelength reference is obtained:

1. Use telluric lines in Earth's atmosphere (e.g., O_2 at 6300 Å) as reference

Advantages:

- simple & inexpensive
- feasible for almost any high-resolution spectrograph
- both the reference and stellar light illuminate the spectrograph in the same manner and are recorded at the same time ⇒ minimization of instrumental shifts
- RV precision of a few tens of m/s achievable

Disadvantages:

- limited wavelength coverage (10s of Å) ⇒ small number of spectral lines can be used for the RV measurements
- variations of Earth's atmosphere (temperature and pressure changes, winds)
- line depths of telluric lines vary with air mass
- a star cannot be observed without telluric lines which is needed in the reduction process

2. Hydrogen-Fluoride (HF) absorption cell

Advantages:
- stable reference, no temperature, pressure, or wind shifts
- RV precision of $\approx$10 m/s

Disadvantages:
- long path length is needed (1 m) to produce suitable absorption lines, problem if space is limited in the spectrograph
- HF is sensitive to pressure shifts
- provides absorption features over only a narrow wavelength range in the optical (100 Å)
- has to be refilled with each observing run, dangerous gas, so safety concerns are a real issue

Problem 4.3 *(1 point)*

The stellar radius is given by

$$R_* = 0.55 \frac{\tau}{1\text{ hr}} \left(\frac{M_*}{M_\odot} \frac{1\text{ hr}}{P} \right)^{1/3} R_\odot.$$

Reading off the transit duration from the graph yields $\tau = 0.3 \ldots 0.4$ days. Plugging in the values, we obtain

$$R = 1.7 \ldots 2.2\ R_\odot.$$

This is definitely not a solar-type star. Most likely it is not a good planet candidate. But we have more information: the transit depth. Since $(R_\mathrm{p}/R_*)^2 = 0.011$, the planet has a radius

$$R_\mathrm{p} = 0.18 \ldots 0.23\ R_\odot = 1.8 \ldots 2.3\ R_\mathrm{Jup},$$

which is too big for a planet. The chances are high that the companion is an M dwarf star.

Problem 4.4 *(3 points)*

Assume power laws

$$c_s^2 \propto T \propto r^{-\xi} \qquad \text{and} \qquad \Sigma \propto r^{-\zeta}, \tag{1}$$

so that

$$\nu = \alpha \frac{c_s^2}{\Omega_K} \propto r^{-\xi+3/2}. \tag{2}$$

Substitute these into the formula for the radial velocity

$$v_r = -\frac{3\nu}{2r} - \frac{3}{\Sigma} \frac{\partial}{\partial r}(\Sigma\nu) = -\frac{3}{\Sigma\sqrt{r}} \frac{\partial}{\partial r} \left(\Sigma\nu\sqrt{r}\right) \tag{3}$$

to get

$$v_r \propto \frac{\nu}{r} \propto r^{-\xi+1/2} \tag{4}$$

Now, the stationary continuity equation,

$$\frac{\partial(\Sigma r v_r)}{\partial r} = 0, \tag{5}$$

requires $\Sigma r v_r = \text{const}$ or

$$r^{-\zeta} \cdot r \cdot r^{-\xi+1/2} = r^{-\zeta} \cdot r^{-\xi+3/2} = \text{const}, \tag{6}$$

whence

$$\zeta = -\xi + 3/2. \tag{7}$$

Therefore, a general solution is

$$T \propto r^{-\xi}, \qquad \nu \propto r^{-\xi+3/2}, \qquad \Sigma \propto r^{\xi-3/2}. \tag{8}$$

To be "physical", these solutions must have at least $\xi > 0$ (the farther out from the star, the colder). On the other hand, $\xi < 3/2$ is a reasonable requirement because the surface density is not expected to grow outward. These limitations are not strict though.

Note that a steepening temperature profile ($\sigma \uparrow$) results in a shallower density profile and vice versa. This can be understood by looking again at the continuity criterion. The product rv_r is roughly proportional to ν and thus to c_s^2 and T. Hence the product ΣT is conserved in the stationary case. The profiles of the two quantities must therefore compensate.

Plotting several of these solutions, for instance for $\xi = 0$, $1/2$, 1, and $3/2$ is straightforward. Hopefully you will not do that in linear scale ... log-log is the most natural scale to plot power laws.