Physics of Planetary Systems — Exercises
Suggested Solutions to Set 3

Problem 3.1 *(3 points)*

The provided Python script creates a Lomb-Scargle periodogram (Fig. 1) and finds a peak amplitude at a period

$$P = 4.23 \text{ d}. \tag{1}$$

Fitting the phase-folded RV data with a sinusoid (Fig. 2),

$$RV(\varphi) = v_{\text{system}} + K_1 \sin(2\pi\phi + \Delta), \tag{2}$$

results in a system velocity

$$v_{\text{system}} = (8.9 \pm 1.1) \text{ m/s} \tag{3}$$

and an RV (semi-)amplitude

$$K_1 = (55.1 \pm 1.5) \text{ m/s}. \tag{4}$$

With the given stellar mass, $\mathscr{M}_1 = (1.16 \pm 0.05)\mathscr{M}_\odot$, the resulting minimum mass is (assuming $\mathscr{M}_2 \ll \mathscr{M}_1$)

$$\mathscr{M}_2 \sin i = K_1 \sqrt[3]{\frac{P\mathscr{M}_1^2}{2\pi G}} = 9.2 \times 10^{26} \text{ kg} \approx 0.48\ \mathscr{M}_{\text{Jup}}. \tag{5}$$

Bonus: a closer at the right panel of Fig. 2 reveals that the residuals (a) are spread more widely than the error bars would allow and (b) do not quite follow a normal distribution but look rather bimodal. Both effects hint at something systematic. If we now repeat our analysis with the residuals, we find a suspicious period of 1 day (Fig. 3). The intra-day residuals (Fig. 4) look more like a step function. This reflects the fact that the data have been collected with at least two different instruments (or calibrations) at two different geographical longitudes. Such effects could be accounted for in further analysis.

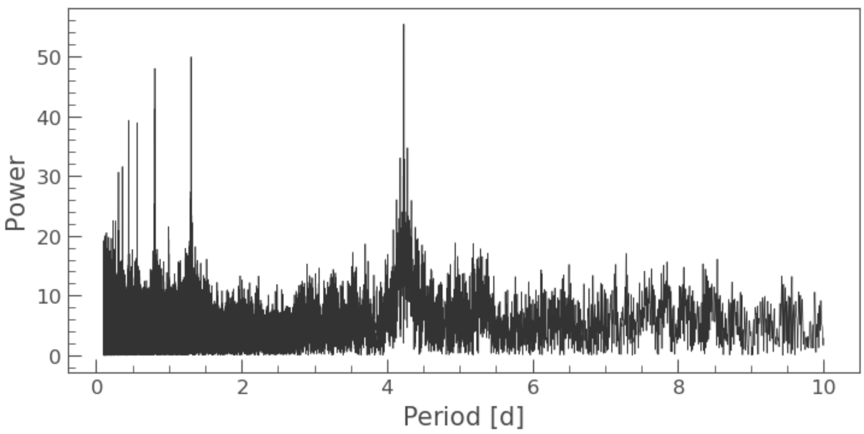

Figure 1: LOMB-SCARGLE periodogram of the original 51 Peg data, showing a peak at a period of 4.23 d.

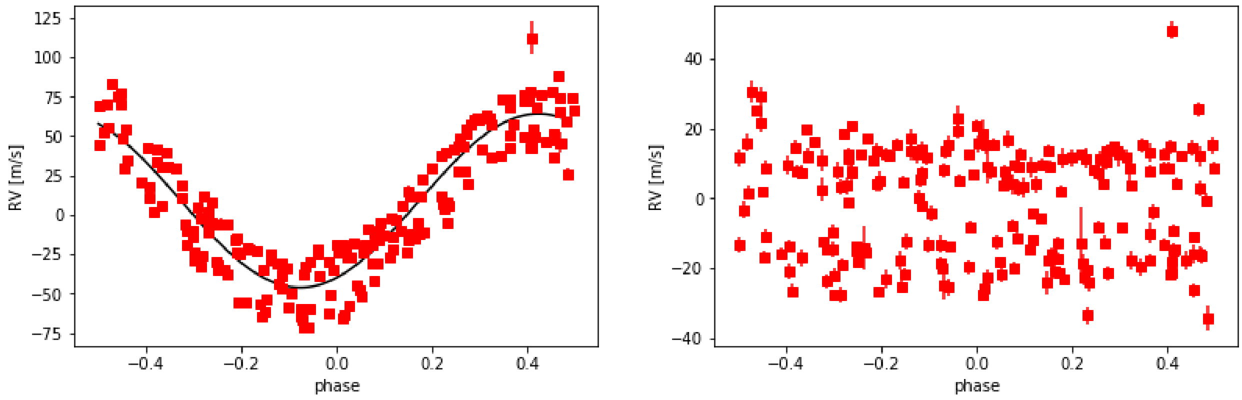

Figure 2: (Left) phase-folded light curve with fitted sinusoidal model and (right) residuals after subtraction of the model.

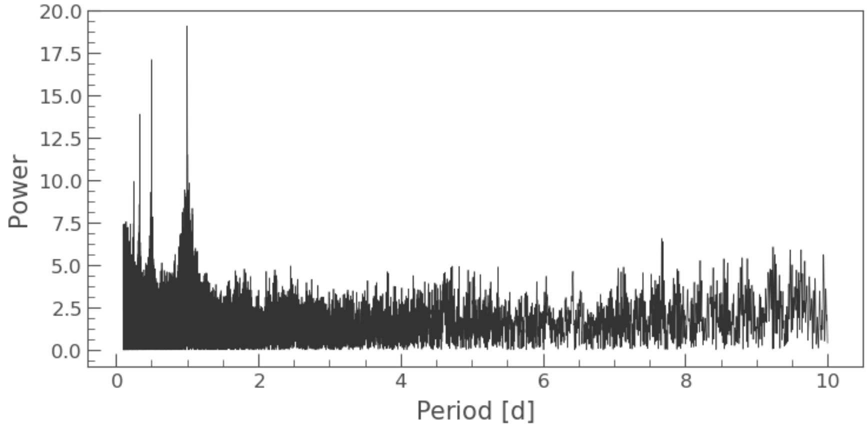

Figure 3: Periodogram of the residuals, showing a peak at a period of 1 d.

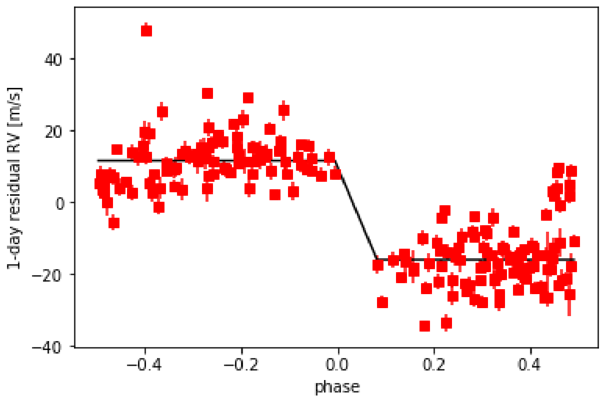

Figure 4: Intra-day residuals, showing a calibration offset between two different observing locations.

Bonus problem 3.2 *(1 extra point)*

Letting $\mathscr{M}_2' \equiv \mathscr{M}_2' \sin i$, errors can be estimated via

$$\begin{aligned}
\Delta\mathscr{M}_2' &\approx \left|\frac{\mathrm{d}\mathscr{M}_2'}{\mathrm{d}K_1}\right|\Delta K_1 + \left|\frac{\mathrm{d}\mathscr{M}_2'}{\mathrm{d}P}\right|\Delta P + \left|\frac{\mathrm{d}\mathscr{M}_2'}{\mathrm{d}\mathscr{M}_1}\right|\Delta\mathscr{M}_1 \\
&= \frac{\mathscr{M}_2'}{K_1}\Delta K_1 + \frac{1}{3}\frac{\mathscr{M}_2'}{P}\Delta P + \frac{2}{3}\frac{\mathscr{M}_2'}{\mathscr{M}_1}\Delta\mathscr{M}_1 \\
\frac{\Delta\mathscr{M}_2'}{\mathscr{M}_2'} &= \underbrace{\frac{\Delta K_1}{K_1}}_{2.8\,\%} + \underbrace{\frac{1}{3}\frac{\Delta P}{P}}_{\text{small}^*} + \underbrace{\frac{2}{3}\frac{\Delta\mathscr{M}_1}{\mathscr{M}_1}}_{2.9\,\%}.
\end{aligned} \tag{6}$$

The statistical error is dominated in equal parts by the stellar mass and the radial velocity amplitude.

* The smallness of the uncertainty in orbital period could be shown by tracing the width of the peak in the periodogram.

Bonus problem 3.3 *(1 extra point)*

Besides the above-mentioned issue with calibration offsets, stellar activity is the usual suspect for additional noise that is not instrumental.

Problem 3.4 *(2 points)*

Taking into account that the angular momentum is conserved, the rotation period of the protosun would be as short as

$$P = \frac{2\pi}{\Omega} = \frac{2\pi R^2}{L} \sim 6 \cdot \frac{(7\cdot 10^{10})^2}{10^{21}} \sim 30\mathrm{s} \sim 0.5 min.$$

On the other hand, the minimum rotation period P_{min}, at which the protosun would not be broken by the centrifugal force, is the orbital period of a Keplerian orbit with radius $R_\odot$. Were the orbital radius equal to 1au, we would have $P_{min} = 1$yr. For the orbital radius of $R_\odot = 1/200$au, the minimum period according to the 3rd Kepler law is

$$P_{min} = 1\mathrm{yr}\cdot\left(\frac{1}{200}\right)^{3/2} \sim \left(\frac{1}{200\cdot 15}\right) \sim \left(\frac{400\cdot 1440}{200\cdot 15}\right) \sim 200 min.$$

Bonus problem 3.5 *(2 extra points)*

Let us begin with a look at the angular momentum that a mass element $\mathrm{d}m = 2\pi r\Sigma\mathrm{d}r$ (see Fig. 5) carries:

$$\mathrm{d}L = l\mathrm{d}m, \quad \text{where } l \equiv r v_\mathrm{K}\ (= r^2\Omega_\mathrm{K}). \tag{7}$$

As the element drifts inward, the specific angular momentum l (i. e. angular momentum per mass) changes according to

$$l' = \frac{\mathrm{d}l}{\mathrm{d}r} = v_\mathrm{K} + r v_\mathrm{K}'. \tag{8}$$

With

$$v_\mathrm{K} = \sqrt{\frac{G\mathscr{M}_*}{r}} \quad \text{and} \quad v_\mathrm{K}' = -\frac{v_\mathrm{K}}{2r}, \tag{9}$$

we obtain

$$l' = v_\mathrm{K} - \frac{v_\mathrm{K}}{2} = \frac{v_\mathrm{K}}{2} = \frac{l}{2r} \quad \text{or} \quad \mathrm{d}l = \frac{v_\mathrm{K}}{2}\mathrm{d}r. \tag{10}$$

For the inward radial drift, where $\mathrm{d}r < 0$, we find $\mathrm{d}l < 0$. That is, each mass element drifting inward loses angular momentum. Because the total angular momentum must be conserved, it must be transferred outward.

However, the question is whether the angular momentum "leaps" outward from mass element to mass element more quickly than the mass drifts inward, i. e. whether angular momentum really moves outward or just migrates more slowly inward. While the full answer will depend on the specific local conditions, the following calculation can shed some light. The total angular momentum inside a given distance r_0 can be written as

$$L_\mathrm{i}(r<r_0) = \int_{r_\mathrm{min}}^{r_0} l\,2\pi r\Sigma\,\mathrm{d}r = \int_0^{m_0} l\,\mathrm{d}m, \tag{11}$$

where $m_0 = m_0(t)$ is the mass contained within $r<r_0$. This quantity changes over time for two reasons: (a) the material that is already in the inner disk (at $r<r_0$) will lose specific momentum, its l will reduce; (b) fresh material will enter the region from outside adding new angular momentum, m_0 changes. The two contributions are given by

$$\tfrac{\mathrm{d}L_\mathrm{i}}{\mathrm{d}t}(r<r_0) = \overbrace{\int_0^{m_0} \underbrace{\frac{\mathrm{d}l}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}}_{=l'v_r<0}\mathrm{d}m}^{\text{(a)}} \;+\; \overbrace{l(r_0)\underbrace{\frac{\mathrm{d}m_0}{\mathrm{d}t}}_{=\dot{m}_0}}^{\text{(b)}} = \int_0^{m_0} l' v_r\,\mathrm{d}m \;+\; l(r_0)\dot{m}_0. \tag{12}$$

In a static disk, where the mass flow is constant, we have $\dot{m}_0 = \dot{M} = -2\pi r\Sigma v_r = \text{const}$ (with $v_r<0$). We can simplify

$$\begin{aligned}\tfrac{\mathrm{d}L_\mathrm{i}}{\mathrm{d}t}(r<r_0) \;&=\; \int_{r_\mathrm{min}}^{r_0} l'\underbrace{v_r 2\pi r\Sigma}_{-\dot{M}=\text{const}}\,\mathrm{d}r \;+\; l(r_0)\dot{M} \;=\; -\dot{M}\int_{r_\mathrm{min}}^{r_0} l'\,\mathrm{d}r \;+\; l(r_0)\dot{M} \;=\; -\dot{M}\left[l(r_0)-l(r_\mathrm{min})\right] \;+\; l(r_0)\dot{M}\\ &=\; \dot{M}l(r_\mathrm{min}) \quad (\geq 0).\end{aligned} \tag{13}$$

where $v_r<0$ still. It may seem odd that the change in total angular momentum is positive and depends on what happens at the *inner* edge, but that is actually what is expected for the static disc: the fresh material coming in from outer region compensates for the loss caused by material slowly drifting inward. And if the inner edge moves inward (while the surface density in the rest of the inner disk stays the same), the total mass *and* angular momentum in the inner disk will actually increase. This effect would be small though because $l \propto \sqrt{r} \to 0$ as $r\to 0$, and hence, $l(r_\mathrm{min}) \ll l(r_0)$ for $r_\mathrm{min} \ll r_0$.

If $\dot{M}(r) \neq \text{const}$, net transport of angular momentum accross r_0 could happen. For example, if $\dot{M}(r)$ were to increase towards the star, more angular momentum would be lost for $r<r_0$ than gained from inflow from the outer region. Given (see lecture)

$$v_r = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma\nu\sqrt{r}\right) = -\frac{3\nu}{2r} - \frac{3}{\Sigma}\frac{\partial}{\partial r}(\Sigma\nu), \tag{14}$$

this is equivalent to the question whether

$$\dot{M} = -2\pi\Sigma r v_r = 6\pi\sqrt{r}\frac{\partial}{\partial r}\left(\Sigma\nu\sqrt{r}\right) = 3\pi\Sigma\nu + 6\pi r\frac{\partial}{\partial r}(\Sigma\nu) \tag{15}$$

increases, is constant, or decreases towards the star:

$$\frac{\partial\dot{M}}{\partial r} \;=\; 3\pi\frac{\partial}{\partial r}(\Sigma\nu) + 6\pi\frac{\partial}{\partial r}(\Sigma\nu) + 6\pi r\frac{\partial^2}{\partial r^2}(\Sigma\nu) \;=\; 9\pi\frac{\partial}{\partial r}(\Sigma\nu) + 6\pi r\frac{\partial^2}{\partial r^2}(\Sigma\nu) \;\overset{?}{\gtreqless}\; 0. \tag{16}$$

The answer depends on the local slope and curvature of $\Sigma\nu$ as a function of r. Again, for a static disk, where $\Sigma\nu = \text{const}$, we find $\dot{M} = \text{const}$, and hence, no net outward angular momentum transport accross r_0.

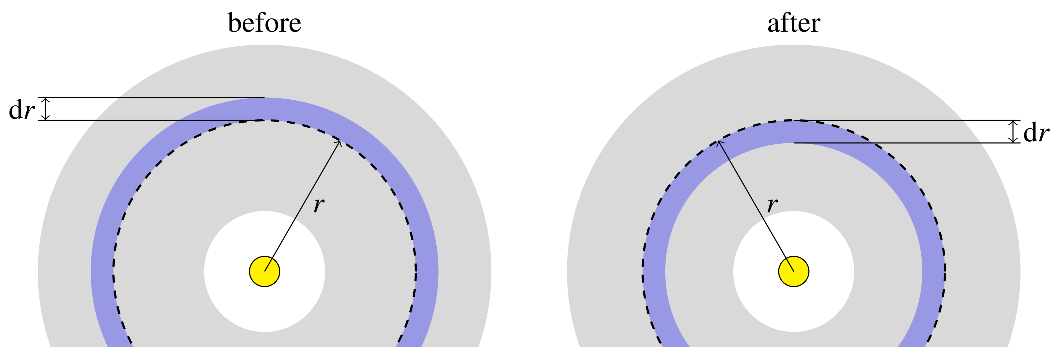

Figure 5: A narrow annulus of width dr drifts inwards, passing the dashed line that marks a distance of r.