Physics of Planetary Systems — Exercises
Suggested Solutions to Set 2

Problem 2.1 *(2 points)*

In the lecture the observed radial velocity amplitude of the primary (star), $K_1 = v_{\rm obs}$, was derived:

$$\underbrace{\frac{\mathscr{M}_2^3 \sin^3 i}{(\mathscr{M}_1 + \mathscr{M}_2)^2}}_{\equiv f} = \frac{K_1^3 P \left(1-e^2\right)^{3/2}}{2\pi G}. \tag{1}$$

where f is the mass function. Hence we have

$$v_{\rm obs} = K_1 = \frac{\sqrt[3]{2\pi G f/P}}{\sqrt{1-e^2}}. \tag{2}$$

Assuming $\mathscr{M}_1 \gg \mathscr{M}_2$ and $e \approx 0$, we get an approximate

$$v_{\rm obs} \approx \sqrt[3]{\frac{2\pi G \mathscr{M}_2^3 \sin^3 i}{P \mathscr{M}_1^2}} = \mathscr{M}_2 \sin i \sqrt[3]{\frac{2\pi G}{P\mathscr{M}_1^2}}. \tag{3}$$

Now we can expand with the suggested units (and replace some variables): $\mathscr{M}_2 = m_{\rm p}$ in Jupiter masses, P in years, and $\mathscr{M}_1 = m_{\rm s}$ in Solar masses:

$$\begin{aligned} v_{\rm obs} &\approx \mathscr{M}_{\rm Jup} \left(\frac{m_{\rm p}}{\mathscr{M}_{\rm Jup}}\right) \sin i \sqrt[3]{\frac{2\pi G/\left(\mathrm{yr}\mathscr{M}_{\rm Sun}^2\right)}{(P/\mathrm{yr})\left(m_{\rm s}/\mathscr{M}_{\rm Sun}\right)^2}} \\ &= \mathscr{M}_{\rm Jup} \sqrt[3]{\frac{2\pi G}{\mathrm{yr}\mathscr{M}_{\rm Sun}^2}} \times \frac{m_{\rm p}[\mathscr{M}_{\rm Jup}] \sin i}{P[\mathrm{yr}]^{1/3} m_{\rm s}[\mathscr{M}_{\rm Sun}]^{2/3}}. \end{aligned} \tag{4}$$

The pre-factor is constant and can be computed directly from its constituents. Alternatively, we can tranform it further, considering that 1 yr is Earth's orbital period:

$$1\ \mathrm{yr} = 2\pi \sqrt{\frac{(1\ \mathrm{au})^3}{G\mathscr{M}_{\rm Sun}}}, \tag{5}$$

from which we obtain

$$\mathscr{M}_{\rm Jup} \sqrt[3]{\frac{2\pi G}{\mathrm{yr}\mathscr{M}_{\rm Sun}^2}} = \frac{\mathscr{M}_{\rm Jup}}{\mathscr{M}_{\rm Sun}} \sqrt[3]{\frac{2\pi G \mathscr{M}_{\rm Sun}}{\mathrm{yr}}} = \frac{\mathscr{M}_{\rm Jup}}{\mathscr{M}_{\rm Sun}} \sqrt{\frac{G\mathscr{M}_{\rm Sun}}{1\ \mathrm{au}}} = \underbrace{\frac{\mathscr{M}_{\rm Jup}}{\mathscr{M}_{\rm Sun}}}_{1/1050} \underbrace{v_{\rm Earth}}_{30\ \mathrm{km/s}} \approx 28.4\ \mathrm{m/s}, \tag{6}$$

and finally,

$$v_{\rm obs}[\mathrm{m/s}] \approx 28.4 \times \frac{m_{\rm p}[\mathscr{M}_{\rm Jup}] \sin i}{P[\mathrm{yr}]^{1/3} m_{\rm s}[\mathscr{M}_{\rm Sun}]^{2/3}}. \tag{7}$$

Problem 2.2 *(2 points)*

Recall the mass function given in the lecture:

$$f(m) = \frac{\left(m_{\rm planet} \sin i\right)^3}{(m_{\rm star} + m_{\rm planet})^2} = \frac{PK^3\left(1-e^2\right)^{3/2}}{2\pi G}.$$

As instructed, we have to consider identical host star mass and orbital inclination for both given cases. Let $m_{\rm c}$ and $m_{\rm e}$ (both $\ll m_{\rm star}$) be the masses of the circular and eccentric planets, and $f_{\rm c}$ and $f_{\rm e}$ their respective mass functions. Taking the ratio of the mass functions and now using that stellar masses and $\sin i$ are the same for both stars, we have

$$\frac{f_{\rm e}}{f_{\rm c}} = \left(\frac{m_{\rm e}}{m_{\rm c}}\right)^3 = \frac{\left(1-e^2\right)^{3/2}}{1} \quad = \quad 0.08,$$

and thus $m_{\rm e} = 0.43\, m_{\rm c}$. So, the planet in the more eccentric orbit has about 40% of the mass of the one in the circular orbit.

Problem 2.3 *(2 points)*

From the virial theorem, the condition for the Jeans radius or mass is $K = |U|/2$, where

$$K = \frac{1}{2}\mathscr{M}v^2 = \frac{1}{2}\frac{3kT}{\mu m_{\rm p}}\mathscr{M} \qquad \text{and} \qquad |U| = \frac{3}{5}\frac{G\mathscr{M}^2}{R}, \tag{8}$$

so that

$$\frac{kT}{\mu m_{\rm p}}\mathscr{M} = \frac{1}{5}\frac{G\mathscr{M}^2}{R}, \tag{9}$$

giving the Jeans radius

$$R_J = \frac{1}{5}\mu m_{\rm p}\frac{G\mathscr{M}}{kT}, \tag{10}$$

which contains an additional factor 1/5 (and μ, of course).

We now re-derive the Jeans mass:

$$\mathscr{M} \sim \mu m_{\rm p}\cdot n\cdot\frac{4}{3}\pi R^3 \qquad \text{or} \qquad R \sim \left(\frac{3}{4}\pi\right)^{1/3}\left(\frac{\mathscr{M}}{n\mu m_{\rm p}}\right)^{1/3} \tag{11}$$

Substituting this into (9) yields

$$\frac{kT}{\mu m_{\rm p}} < \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\mathscr{M}^{2/3}(n\mu m_{\rm p})^{1/3} \tag{12}$$

or

$$kT < \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\mathscr{M}^{2/3}n^{1/3}(\mu m_{\rm p})^{4/3} \tag{13}$$

or

$$\mathscr{M}_{\rm J} = 5^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\left(\frac{kT}{Gn^{1/3}(\mu m_{\rm p})^{4/3}}\right)^{3/2} \tag{14}$$

or

$$\mathscr{M}_{\rm J} = \underbrace{5^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}}_{5.46}\left(\frac{k}{G}\right)^{3/2}\frac{1}{(\mu m_{\rm p})^2}\frac{T^{3/2}}{n^{1/2}}. \tag{15}$$

The difference to the result derived in the lecture is the additional factor $\approx 5.5/\mu^2$, which is actually close to unity for a mix of molecular hydrogen (H_2) with a bit of helium (He) with an effective $\mu \approx 2\ldots 2.5$.

Problem 2.4 *(2 points)*

The potential energy of a homogeneous sphere of uniform density with radius R and mass $\mathscr{M}$ is

$$|U| = \frac{3}{5}\frac{G\mathscr{M}^2}{R}$$

The *total* energy of a spherical cloud if an ideal gas is $E = K + U$ or, explicitly,

$$E = \frac{3}{2}kT\frac{\mathscr{M}}{\mu m_\mathrm{p}} - \frac{3}{5}\frac{G\mathscr{M}^2}{R}.$$

As in the initial state $T \approx 0$, $R \approx \infty$, and hence $E \approx 0$, and the energy is conserved ($E = \mathrm{const}$), for the final state we find

$$\frac{3}{2}kT\frac{\mathscr{M}}{\mu m_\mathrm{p}} = \frac{3}{5}\frac{G\mathscr{M}^2}{R}.$$

and

$$T = \frac{2G\mathscr{M}\mu m_\mathrm{p}}{5kR}$$

With $\mu = 2$, this gives (in cgs units)

$$T \approx \frac{4 \cdot 7 \cdot 10^{-8} \cdot 2 \cdot 10^{33} \cdot 2 \cdot 10^{-24}}{5 \cdot 1.4 \cdot 10^{-16} \cdot 200 \cdot 1.5 \cdot 10^{13}}\ \mathrm{K} \approx 500\ \mathrm{K}.$$

(You obtain similar results when invoking the virial theorem or the Jeans criterion instead, as kinetic and gravitational energy are equated in all three cases, just with slightly different prefactors. However, note that the virial theorem would only apply if we allowed for loss of energy, e. g., due to radiation.)

Problem 2.5 *(1 point)*

Assuming a rather high temperature of, say, 1000 K at 1 au from the Sun gives the sound speed

$$c_\mathrm{s} \approx \sqrt{\frac{kT}{\mu m_\mathrm{p}}} \sim \sqrt{\frac{1.4 \times 10^{-23}\ \mathrm{J/K} \cdot 1000\ \mathrm{K}}{2 \cdot 1.7 \times 10^{-27}\ \mathrm{kg}}} \sim \sqrt{5 \times 10^6 \frac{\mathrm{m}^2}{\mathrm{s}^2}} \sim 2 \times 10^3\ \mathrm{m/s} \sim 2\ \mathrm{km/s}. \tag{16}$$

The Keplerian circular velocity at the same distance from the Sun is given by

$$v_\mathrm{K} = \sqrt{G\mathscr{M}_\star/r} \approx 30\ \mathrm{km/s}, \tag{17}$$

which is much greater than the sound speed. Given that both temperature and Keplerian velocity decrease with increasing distance, the inequality $cs \ll v_\mathrm{K}$ holds as well at 10 au or 100 au.