Physics of Planetary Systems — Exercises
Suggested Solutions to Set 12

Problem 12.1 *(2 points)*

To address the question of how planetary radii and temperatures are related, we need the temperatures. Measuring the temperatures is very difficult or completely unfeasible is most cases, so that we have to rely on estimates based on the typical irradiation from the star, depends on the stellar luminosity, L_*, and the star-planet distance, r. Equating absorbed stellar radiation (over the cross section of the planet) and emitted thermal radiation (over its whole surface), we find

$$\begin{aligned}
L_{\text{in}} &\overset{!}{=} L_{\text{out}} \\
\frac{L_*}{4\pi r^2}\pi R_{\text{p}}^2(1-A) &= 4\pi R_{\text{p}}^2 \sigma T_{\text{p}}^4 \\
\frac{4\pi\sigma R_*^2 T_*^4}{4\pi r^2}\pi R_{\text{p}}^2(1-A) &= 4\pi R_{\text{p}}^2 \sigma T_{\text{p}}^4 \\
\frac{R_*^2 T_*^4}{r^2}(1-A) &= 4T_{\text{p}}^4 \\
\sqrt{\frac{R_*}{2r}}\sqrt[4]{1-A}\ T_* &= T_{\text{p}}, \qquad (1)
\end{aligned}$$

where R_* and T_* are the stellar radius and temperature, respectively, and A the planet's Bond albedo. In solar units, we have

$$T_{\text{p}} = 279\ \text{K} \times T_*[T_\odot]\sqrt{\frac{R_*[R_\odot]}{2r[\text{au}]}}, \qquad (2)$$

where we assumed $\sqrt[4]{1-A} \approx 1$. After extracting the tabulated values for M_{p}, R_{p}, R_*, $T_*(= T_{\text{eff}})$, and r(assumed $\approx a$) from `https://exoplanet.eu/`, we can filter for $M_{\text{p}} > 0.3M_{\text{Jup}}$, calculate T_{p} and plot T_{p} vs R_{p}.

The resulting distribution is shown in Fig. 1. For low temperatures ($T_{\text{p}} \lesssim 1300$ K), all planets have their radii close to the Jupiter radius, which is expected from electron degeneracy pressure in equilibrium with gravity. For higher temperatures ($T_{\text{p}} \gtrsim 1300$ K) the extra heat from the stars puffs the outer planet envelopes up, leading to drasticly increased radii.

Extra info: we could have made our lives a bit easier by plotting T_{p} vs R_{p} directly on the `exoplanet` website because the database already provides the calculcated temperatures.

Problem 12.2 *(1 point)*

In the lecture, the connection between planets and host star metallicity was given, originally derived by Valenti & Fischer. For a metallicity of [Fe/H] = 0.5, the frequency of planet hosting stars is $25\,\% \leq p \leq 30\,\%$ (Fig. 2). So, in a sample of $n = 100$ stars one would expect to find $25 \leq np \leq 30$ stars with planets. If we ask for the probability to find at least 30 planetary systems, we have to use the binomial distribution $B(n, p, k)$, where k denotes the number of planetary systems:

$$P(X > 29) = \sum_{k=30}^{100} B(100, 0.25\ldots0.3, k) = 1 - \sum_{k=0}^{29} B(100, 0.25\ldots0.3, k) = 0.15\ldots0.55.$$

Thus, we would expect to find more than 29 planets with a probability of 15–55 %. Note that the probability in the cumulative value comes from a number close to 30. The probability to find e. g. 70 planets is almost zero.

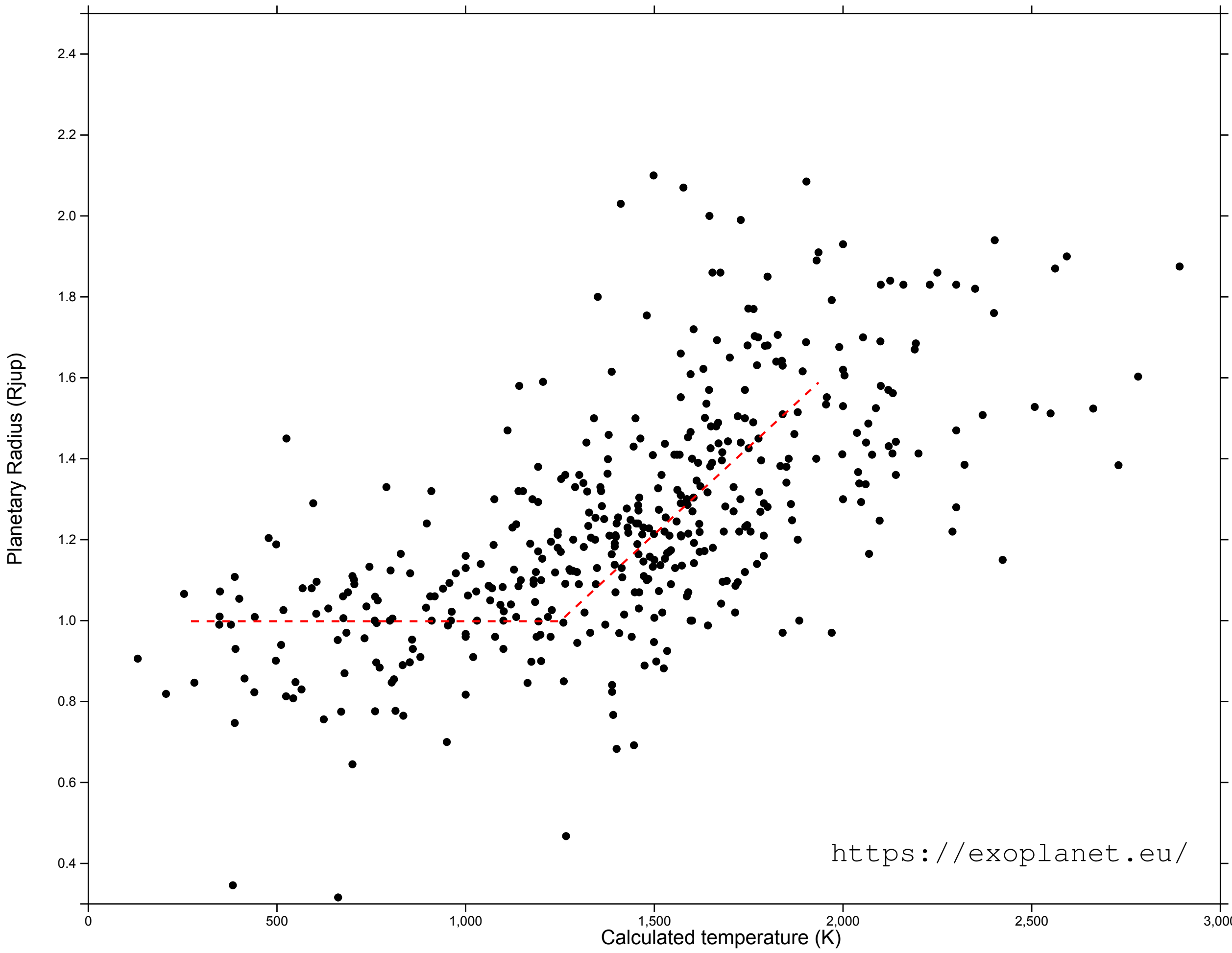

Figure 1: Radii vs calculated temperatures of confirmed planets with masses $M_\mathrm{p} > 0.3 M_\mathrm{Jup}$. Dashed lines indicate the two rough temperature regimes.

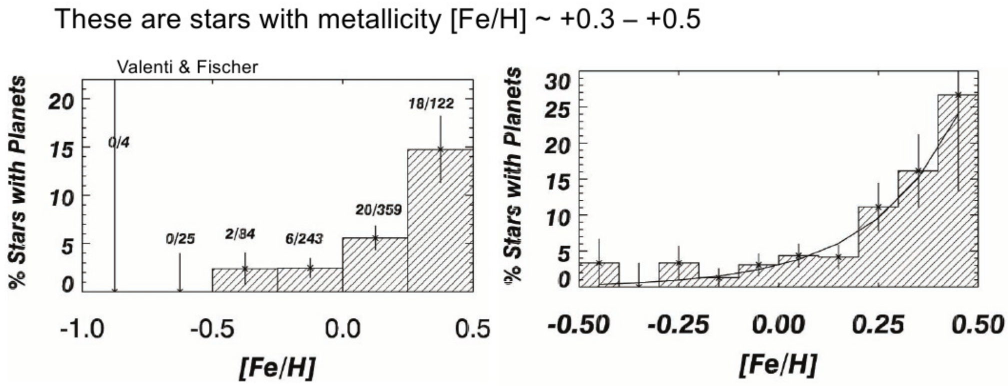

Figure 2: The increasing trend in the fraction of stars with planets as a function of metallicity (Valentin & Fischer). *Right*: Same as *left*, but divided into 0.1 dex metallicity bins. The trend is fitted with a power law, yielding the probability for giant planets: $P = 0.03[(N_{\rm Fe}/N_{\rm H})/(N_{\rm Fe}/N_{\rm H})_{\odot}]^2$

Problem 12.3 *(2 points)*

From energy conservation,

$$E_{\rm kin} + E_{\rm pot} = {\rm const}, \tag{3}$$

and

$$E_{\rm kin} = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega\, r)^2 = \frac{2\pi^2}{P^2}mr^2 = \frac{2\pi^2}{\frac{4\pi^2}{G\mathscr{M}}r^3}mr^2 = \frac{G\mathscr{M}m}{2r}, \tag{4}$$

in combination with the virial theorem,

$$-2E_{\rm kin} = E_{\rm pot}, \tag{5}$$

we find

$$\overbrace{-\frac{G\mathscr{M}_*\mathscr{M}_{\rm N}}{2r_{\rm N,0}} + \left(-\frac{G\mathscr{M}_*\mathscr{M}_{\rm U}}{2r_{\rm U,0}}\right) + \left(-\frac{G\mathscr{M}_*\mathscr{M}_{\rm J}}{2r_{\rm J,0}}\right)}^{\text{before the scattering event}} = \overbrace{-\frac{G\mathscr{M}_*\mathscr{M}_{\rm N}}{2r_{\rm N,1}} + \left(-\frac{G\mathscr{M}_*\mathscr{M}_{\rm U}}{2r_{\rm U,1}}\right) + \left(-\frac{G\mathscr{M}_*\mathscr{M}_{\rm J}}{2r_{\rm J,1}}\right)}^{\text{after the scattering event}}, \tag{6}$$

and therefore,

$$r_{\rm J,0} = \mathscr{M}_{\rm J}\left[\frac{\mathscr{M}_{\rm N}}{r_{\rm N,1}} - \frac{\mathscr{M}_{\rm N}}{r_{\rm N,0}} + \frac{\mathscr{M}_{\rm U}}{r_{\rm U,1}} - \frac{\mathscr{M}_{\rm U}}{r_{\rm U,0}} + \frac{\mathscr{M}_{\rm J}}{r_{\rm J,1}}\right]^{-1} = r_{\rm J,1}\left[1 - \frac{\mathscr{M}_{\rm N}}{\mathscr{M}_{\rm J}}\frac{\Delta r_{\rm N}}{r_{\rm N,0}}\frac{r_{\rm J,1}}{r_{\rm N,1}} - \frac{\mathscr{M}_{\rm U}}{\mathscr{M}_{\rm J}}\frac{\Delta r_{\rm U}}{r_{\rm U,0}}\frac{r_{\rm J,1}}{r_{\rm U,1}}\right]^{-1}. \tag{7}$$

With $r_{\rm U,0} = r_{\rm N,0} = 7$ au, $r_{\rm U,1} = 19$ au, $r_{\rm N,1} = 30$ au, $r_{\rm J,1} = 5.2$ au, $\mathscr{M}_{\rm N} = 17\,\mathscr{M}_{\oplus}$, $\mathscr{M}_{\rm U} = 14\,\mathscr{M}_{\oplus}$, and $\mathscr{M}_{\rm J} = 318\,\mathscr{M}_{\oplus}$, the result is

$$r_{\rm J,0} = 5.48 \text{ au}, \quad \Delta r_{\rm J} = r_{\rm J,1} - r_{\rm J,0} = -0.28 \text{ au}, \quad \frac{\Delta r_{\rm J}}{r_{\rm J,0}} = -5\,\%. \tag{8}$$

Alternatively, from angular momentum conservation,

$$L = mr^2\omega = 2\pi r^2\frac{m}{P} = 2\pi r^2\frac{m}{\sqrt{\frac{4\pi^2r^3}{G\mathscr{M}}}} = m\sqrt{\mathscr{M}Gr}, \tag{9}$$

we find

$$\begin{aligned}&\sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm N}\sqrt{r_{\rm N,0}} + \sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm U}\sqrt{r_{\rm U,0}} + \sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm J}\sqrt{r_{\rm J,0}}\\ &= \sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm N}\sqrt{r_{\rm N,1}} + \sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm U}\sqrt{r_{\rm U,1}} + \sqrt{\mathscr{M}_*G}\mathscr{M}_{\rm J}\sqrt{r_{\rm J,1}}\end{aligned} \tag{10}$$

and, therefore,

$$
\begin{aligned}
r_{J,0} &= \left(\frac{\mathscr{M}_N}{\mathscr{M}_J}\sqrt{r_{N,1}} + \frac{\mathscr{M}_U}{\mathscr{M}_J}\sqrt{r_{U,1}} + \sqrt{r_{J,1}} - \frac{\mathscr{M}_N}{\mathscr{M}_J}\sqrt{r_{N,0}} - \frac{\mathscr{M}_U}{\mathscr{M}_J}\sqrt{r_{U,0}}\right)^2 \\
&= r_{J,1}\left[1 + \frac{\mathscr{M}_N}{\mathscr{M}_J}\left(\sqrt{\frac{r_{N,1}}{r_{J,1}}} - \sqrt{\frac{r_{N,0}}{r_{J,1}}}\right) + \frac{\mathscr{M}_U}{\mathscr{M}_J}\left(\sqrt{\frac{r_{U,1}}{r_{J,1}}} - \sqrt{\frac{r_{U,0}}{r_{J,1}}}\right)\right]^2 \qquad (11)
\end{aligned}
$$

Here, the result is

$$r_{J,0} = 6.29\ \text{au}, \quad \Delta r_J = 1.09\ \text{au}. \qquad (12)$$

As expected, the change is in any case only slight because Jupiter is much more massive than both Uranus and Neptune.

Extra info: As $|\Delta r_J/r_J|$ is small for both cases, we can approximate the two results further. For energy conservation, we find

$$r_{J,0} \approx r_{J,1}\left(1 + \frac{\mathscr{M}_N}{\mathscr{M}_J}\frac{\Delta r_N}{r_{N,0}}\frac{r_{J,1}}{r_{N,1}} + \frac{\mathscr{M}_U}{\mathscr{M}_J}\frac{\Delta r_U}{r_{U,0}}\frac{r_{J,1}}{r_{U,1}}\right) \qquad (13)$$

because $(1-x)^{-1} \approx 1+x$ for $x \ll 1$. For the other extreme of angular momentum conservation, the approximate result is

$$r_{J,0} \approx r_{J,1}\left[1 + 2\frac{\mathscr{M}_N}{\mathscr{M}_J}\left(\sqrt{\frac{r_{N,1}}{r_{J,1}}} - \sqrt{\frac{r_{N,0}}{r_{J,1}}}\right) + 2\frac{\mathscr{M}_U}{\mathscr{M}_J}\left(\sqrt{\frac{r_{U,1}}{r_{J,1}}} - \sqrt{\frac{r_{U,0}}{r_{J,1}}}\right)\right] \qquad (14)$$

because $(1+x+y)^2 \approx 1+2x+2y$ for $x \ll 1$ and $y \ll 1$. The square and the square root dependence on the distances both create an additional factor of two each, such that the change in Jupiter's distance is roughly four times as great compared to the case of energy conservation. <u>If</u> we had $\Delta r_N/r_{N,1} \ll 1$ and $\Delta r_U/r_{U,1} \ll 1$ (which is not the case!) we could simplify further:

$$r_{J,0} \approx r_{J,1}\left(1 + 4\frac{\mathscr{M}_N}{\mathscr{M}_J}\frac{\Delta r_N}{r_{N,0}}\sqrt{\frac{r_{N,0}}{r_{J,1}}} + 4\frac{\mathscr{M}_U}{\mathscr{M}_J}\frac{\Delta r_U}{r_{U,0}}\sqrt{\frac{r_{U,0}}{r_{J,1}}}\right). \qquad (15)$$

More extra info: The system that we consider is actually an open system because we only look at the three planets while neglecting other players, namely Saturn and a host of planetesimals. These others players receive both energy and angular momentum while interacting with (mostly) Uranus and Neptune. Angular momentum and energy would be perfectly conserved when considering the full system but not in our reduced system. The solution that we get here is therefore not the full solution. Actually, without the other objects involved, the change of Uranus's and Neptune's position would not have happened the way it did. Without the dampening effect of the outer planetesimals, they could not have reached today's safe, almost circular orbits (which don't cross those of Jupiter and/or Saturn anymore). Instead they would have undergone repeated close encounters, getting ejected out of the Solar system eventually.

Bonus problem 12.4

(0.5 extra points for each item)

Here is an incomplete list of (somewhat) open problems:

1. Do planets form in a "standard" way or through gravitational instabilities?
2. What are typical masses of gaseous disks — about MMSN or much larger?
3. How large is Shakura-Sunyaev's α in protoplanetary disks?
4. What is the role of dead zones, does episodic accretion occur?
5. What are the mechanisms of disk dispersal in $\sim 10^7$yr?
6. Does the massive midplane dust layer form?
7. How efficient is sticking at micrometer to millimeter sizes?
8. (Why do meter-sized planetesimals survive fast inward drift in a gas disk?)

9. (What causes planetesimals to grow from meter to kilometer sizes?)
10. How long did gas accretion of Jupiter and Saturn take?
11. Do pulsational instabilities during gas envelope growth occur in reality?
12. Is it true that Uranus and Neptune formed in the Jupiter–Saturn region and then migrated?
13. Are masses and orbital spacing of terrestrial planets rather chance quantities?
14. What is the origin of water on Earth?
15. What allows sub-Earth mass embryos to survive fast type-I migration?
16. What stops migration of “hot Jupiters” near the star?
17. Why didn’t Jupiter and Saturn in our Solar System migrate, or did they?
18. How to explain large orbital eccentricities of many extrasolar planets?
19. What mechanisms clean up planetesimal disks at later stages?
20. (Was there a *Late* Heavy Bombardment in the Solar System?)

and so on . . .