Physics of Planetary Systems — Exercises
Suggested Solutions to Set 11

Problem 11.1 *(2 points)*

We first need to compute the Einstein Radius, θ_E,

$$\begin{aligned}\theta_E &= \sqrt{\frac{4G\mathscr{M}}{c^2}\frac{D_{LS}}{D_L D_S}} = 1.38\times10^{-8}\ \text{rad}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_{LS}[\text{kpc}]}{D_L[\text{kpc}]\ D_S[\text{kpc}]}}\\ &= 2.85\ \text{mas}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_{LS}[\text{kpc}]}{D_L[\text{kpc}]\ D_S[\text{kpc}]}}\end{aligned} \tag{1}$$

We then need to calculate the magnification from:

$$\mu = \frac{u^2+2}{u\sqrt{u^2+4}},$$

where u is defined as $u \equiv \beta/\theta_E$, and β is the impact parameter in radians. Asymptotically, μ can be approximated as

$$\mu \to \begin{cases} 1+\frac{2}{u^4} \text{ for } u \gg 1, \\ \frac{1}{u} \text{ for } u \ll 1. \end{cases} \tag{2}$$

The duration of the event is given by

$$t = \frac{R_E}{v} = \frac{\theta_E D_L}{v},$$

with $R_E = \theta_E D_L$ being the the projected Einstein Radius. The involved distances are $D_L = 2$ kpc, $D_S = 10$ kpc, $D_{LS} = 8$ kpc.

a) Assuming $\mathscr{M} = 1\ \mathscr{M}_\odot$, we obtain $\theta_E = 1.8$ mas, $u = 0.01/1.8 = 0.00552$, and thus, $\underline{\mu = 181}$. With $R_E = \theta_E D_L = 5.4\times10^{13}$ cm and an assumed velocity $v \approx 200$ km/s, the transit duration is $\underline{t = 31.2\ \text{d}}$.

b) $\mathscr{M} = 1\ \mathscr{M}_{\text{Jup}}$ leads to: $\theta_E = 0.0556$ mas, $u = 0.01/0.0556 = 0.18$, $\underline{\mu = 5.63}$, $R_E = 1.66\times10^{12}$ cm, $\underline{t = 23.1\ \text{h}}$.

c) $\mathscr{M} = 1\ \mathscr{M}_\oplus$ leads to: $\theta_E = 1.5\times10^{-11}$ rad $= 0.00312$ mas, $u = 0.01/0.00312 = 3.2$, $\underline{\mu = 1.013}$, $R_E = 9.24\times10^{10}$ cm, $\underline{t = 1.3\ \text{h}}$.

Problem 11.2 *(2 points)*

a: If no companion were present, we would expect the pulses to arrive at times $t'_0 + nP'$, i. e. with constant intervals P' (except for the effect discussed in (**b**)). However the radial displacement (Δr) causes a variation in light travel times, and hence, the pulse timings from those expected, $\Delta t'_n$. The maximum (semi-)amplitudes of radial displacement and timing variations are related to the semi-major axis of the pulsar's orbit, a_{pulsar}, via:

$$\Delta t'_{\max} = \frac{\Delta r_{\max}}{c} = \frac{a_{\text{pulsar}}\sin i}{c}, \tag{3}$$

where i is the inclination of the pulsar–companion orbit relative to the plane of the sky. Using Kepler's third law,

$$P_{\text{orb}} = 2\pi\sqrt{\frac{a^3}{G\mathscr{M}}} \tag{4}$$

with $\mathcal{M} = \mathcal{M}_{\text{pulsar}} + \mathcal{M}_{\text{comp}}$, and the definition of the barycenter,

$$a = a_{\text{pulsar}} + a_{\text{comp}} = a_{\text{pulsar}} \left(1 + \frac{\mathcal{M}_{\text{pulsar}}}{\mathcal{M}_{\text{comp}}}\right), \tag{5}$$

we obtain

$$P_{\text{orb}} = 2\pi \sqrt{\frac{a_{\text{pulsar}}^3 \left(1 + \frac{\mathcal{M}_{\text{pulsar}}}{\mathcal{M}_{\text{comp}}}\right)^3}{G(\mathcal{M}_{\text{pulsar}} + \mathcal{M}_{\text{comp}})}} \approx 2\pi \sqrt{\frac{a_{\text{pulsar}}^3 \mathcal{M}_{\text{pulsar}}^2}{G \mathcal{M}_{\text{comp}}^3}} \overset{\text{eq. (3)}}{=} 2\pi \sqrt{\frac{(c\,\Delta t'_{\max})^3\, \mathcal{M}_{\text{pulsar}}^2}{G\left(\mathcal{M}_{\text{comp}} \sin i\right)^3}}, \tag{6}$$

and after solving for the minimum mass,

$$\mathcal{M}_{\text{comp}} \sin i = c\,\Delta t_{\max} \sqrt[3]{\frac{1}{G}\left(\frac{2\pi \mathcal{M}_{\text{pulsar}}}{P_{\text{orb}}}\right)^2}. \tag{7}$$

Assuming $\mathcal{M}_{\text{pulsar}} = 1.4\ \mathcal{M}_\odot$, $P_{\text{orb}} = 1$ yr, and $\Delta t'_{\max} = 1$ ms, we find

$$\mathcal{M}_{\text{comp}} \sin i = 5 \times 10^{24}\ \text{kg} \approx 0.8\, \mathcal{M}_{\text{Earth}}, \tag{8}$$

i. e. we may have detected an Earth-mass planet (if the inclination is not too far away from 90°).

b: Neglecting relativistic effects, the true times at which the pulsar emits its pulses are given by

$$t_n = t_0 + nP. \tag{9}$$

In contrast, the times at which the pulses arrive at the barycenter of the solar system are

$$t'_n = t_n + \frac{r(t_n)}{c}, \tag{10}$$

where $r(t_n)$ is the distance at the time of pulse emission and c the speed of light. The radial velocity of the pulsar is composed of the (near-constant) system velocity and the variation due to orbital motion around the barycenter:

$$\dot{r} = v_{\text{r}} = v_{\text{sys}} + \Delta v_{\text{r}}(t). \tag{11}$$

The resulting distance is

$$r = r_0 + \int_{t_0}^{t} v_{\text{r}} \text{d}t \ = \ r_0 + (t - t_0) v_{\text{sys}} + \Delta r(t), \tag{12}$$

where Δr is the pulsar's distance from the common barycenter with its companion. Hence we find

$$\begin{aligned} t'_n &= t_n + \frac{r_0 + (t_n - t_0) v_{\text{sys}} + \Delta r(t_n)}{c} \\ &\overset{\text{eq. (9)}}{=} t_0 + nP + \frac{r_0 + nP v_{\text{sys}} + \Delta r(t_n)}{c} \end{aligned} \tag{13}$$

The difference between this and the normal case are the terms $\frac{nPv_{\text{sys}}}{c}$ and $\frac{\Delta r(t_n)}{c}$. If we assume $n = 1$, $v_{\text{sys}} = 200\,\frac{\text{km}}{\text{s}}$, and $\Delta r(t_n) = 1\,\text{au}$, we get $\frac{nPv_{\text{sys}}}{c} \approx 5.8\,\text{h}$ and $\frac{\Delta r(t_n)}{c} \approx 500\,\text{s}$. So even without a companion, the signal arrival time might increase/decrease by up to 5.8 hours each time and vary by up to 500 seconds.

Problem 11.3 *(1 point)*

As discussed in the lecture, the migration rate for type-II migration in the low-mass and high-mass regimes is given by

$$\dot{a}_\mathrm{p} \approx \begin{cases} v_r & \text{(low mass)} \\ v_r\sqrt{\frac{4\pi\Sigma a_\mathrm{p}^2}{\mathscr{M}_\mathrm{p}}} & \text{(high mass),} \end{cases} \tag{14}$$

where Σ is the surface mass density of the surrounding gas disk, v_r its radial drift speed, and a_p and $\mathscr{M}_\mathrm{p}$ the planetary semi-major axis and mass, respectively. (The corresponding timescale could be defined as $t_\mathrm{II} \equiv a_\mathrm{p}/\dot{a}_\mathrm{p}$.) The two regimes are joined where the square-root term becomes unity, i. e.

$$\mathscr{M}_\mathrm{p} = 4\pi\Sigma a_\mathrm{p}^2. \tag{15}$$

Below that critical mass, the planet is less massive than the surrounding disk. It will be dragged along with the viscous gas that slowly drifts toward the star. Above the critical mass, the planet is more massive than the disk and will not be affected as strongly, resulting in a slower migration.

Assuming $a_\mathrm{p} = 1$ au $= 1.5 \times 10^{13}$ cm and $\Sigma = 1000$ g cm^{-2}, we find

$$\mathscr{M}_\mathrm{p} \approx 2.8 \times 10^{30}\ \mathrm{g} \approx 1.5\ \mathscr{M}_\mathrm{Jup}, \tag{16}$$

which is quite a lot.

Bonus problem 11.4 *(3 extra points)*

If the material from both sides of the gap can reach the gap center within one orbital period, the gap is closed. Thus, we have to equate the time the planet needs to reach the same position relative to the gas again with the time needed for the gas to traverse a distance equal to the planet's Hill radius r_H:

$$\frac{2\pi r}{v_\mathrm{rel}} = \frac{r_\mathrm{H}}{v_\mathrm{fill}}, \tag{17}$$

where v_fill is the speed at which the gas can refill the gap and v_rel is the difference between the tangential velocities of gas and embryo:

$$v_\mathrm{rel} = v_\mathrm{K} - v_\mathrm{gas} = v_\mathrm{K}\left(1 - \sqrt{1-2\eta}\right) \approx \eta v_\mathrm{K},$$

with $\eta \equiv c_\mathrm{s}^2/v_\mathrm{K}^2$. Using

$$r_\mathrm{H} = r\left(\frac{\mathscr{M}_\mathrm{p}}{3\mathscr{M}_*}\right)^{1/3}, \tag{18}$$

we arive at

$$\frac{2\pi r}{v_\mathrm{rel}} = \frac{r}{v_\mathrm{fill}}\left(\frac{\mathscr{M}_\mathrm{p}}{3\mathscr{M}_*}\right)^{1/3},$$

and solving for $\mathscr{M}_\mathrm{p}$ leads to

$$\mathscr{M}_\mathrm{p} = 3\mathscr{M}_*\left(\frac{2\pi v_\mathrm{fill}}{v_\mathrm{rel}}\right)^3.$$

Now, we can assume a filling velocity that equals the radial drift velocity:

$$v_\mathrm{fill} = \frac{3\nu}{2r} = \frac{3\alpha c_\mathrm{s}^2}{2r\Omega_\mathrm{K}} = \frac{3\alpha c_\mathrm{s}^2}{2v_\mathrm{K}}, \tag{19}$$

from which we obtain

$$\mathscr{M}_{\rm p} = 3\mathscr{M}_* \left(\frac{3\pi\alpha c_{\rm s}^2}{\eta v_{\rm K}^2}\right)^3 = 3\mathscr{M}_* \left(3\pi\alpha\right)^3 \approx 2500\alpha^3\mathscr{M}_*.$$

Assuming $\alpha = 0.001\ldots0.01$ and $\mathscr{M}_* = M_\odot$ leads to

$$\begin{aligned}\mathscr{M}_{\rm p} &= 2.5\times10^{-6}\mathscr{M}_*\ldots2.5\times10^{-3}\mathscr{M}_* \\ &\approx 1\,M_\oplus\ldots2\,M_{\rm Jup}.\end{aligned}$$