Physics of Planetary Systems — Exercises
Suggested Solutions to Set 10

Problem 10.1 *(2 points)*

We first need to compute the Einstein Radius, θ_E,

$$\begin{aligned}\theta_E &= \sqrt{\frac{4G\mathscr{M}}{c^2}\frac{D_{LS}}{D_L D_S}} = 1.38\times 10^{-8}\ \text{rad}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_{LS}[\text{kpc}]}{D_L[\text{kpc}]\ D_S[\text{kpc}]}} \\ &= 2.85\ \text{mas}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_{LS}[\text{kpc}]}{D_L[\text{kpc}]\ D_S[\text{kpc}]}}\end{aligned} \tag{1}$$

We then need to calculate the magnification from:

$$\mu = \frac{u^2+2}{u\sqrt{u^2+4}},$$

where u is defined as $u \equiv \beta/\theta_E$, and β is the impact parameter in radians. Asymptotically, μ can be approximated as

$$\mu \to \begin{cases} 1+\frac{2}{u^4} \text{ for } u \gg 1, \\ \frac{1}{u} \text{ for } u \ll 1. \end{cases} \tag{2}$$

The duration of the event is given by

$$t = \frac{R_E}{v} = \frac{\theta_E D_L}{v},$$

with $R_E = \theta_E D_L$ being the the projected Einstein Radius. The involved distances are $D_L = 2$ kpc, $D_S = 10$ kpc, $D_{LS} = 8$ kpc.

a) Assuming $\mathscr{M} = 1\ \mathscr{M}_\odot$, we obtain $\theta_E = 1.8$ mas, $u = 0.01/1.8 = 0.00552$, and thus, $\underline{\mu = 181}$. With $R_E = \theta_E D_L = 5.4\times 10^{13}$ cm and an assumed velocity $v \approx 200$ km/s, the transit duration is $\underline{t = 31.2\ \text{d}}$.

b) $\mathscr{M} = 1\ \mathscr{M}_{Jup}$ leads to: $\theta_E = 0.0556$ mas, $u = 0.01/0.0556 = 0.18$, $\underline{\mu = 5.63}$, $R_E = 1.66\times 10^{12}$ cm, $\underline{t = 23.1\ \text{h}}$.

c) $\mathscr{M} = 1\ \mathscr{M}_\oplus$ leads to: $\theta_E = 1.5\times 10^{-11}$ rad $= 0.00312$ mas, $u = 0.01/0.00312 = 3.2$, $\underline{\mu = 1.013}$, $R_E = 9.24\times 10^{10}$ cm, $\underline{t = 1.3\ \text{h}}$.

Problem 10.2 *(2 points)*

Imagine you measure the arrival times of pulses from a pulsar (with $\mathscr{M}_* = 1.4\ \mathscr{M}_\odot$) and you note that the times deviate periodically (with a period $P = 1$ yr) by up to ± 1 ms from those expected for constant intervals. What is the minimum mass of a possible companion that could cause this deviation. *Hint: assume a circular orbit.*

Neglecting relativistic effects, the true times at which the pulsar emits its pulses are given by

$$t_n = t_0 + nP. \tag{3}$$

In contrast, the times at which the pulses arrive at the barycenter of the solar system are

$$t'_n = t_n + \frac{r(t_n)}{c}, \tag{4}$$

where $r(t_n)$ is the distance at the time of pulse emission and c the speed of light. The radial velocity of the pulsar is composed of the (near-constant) system velocity and the variation due to orbital motion around the barycenter:

$$\dot{r} = v_\mathrm{r} = v_\mathrm{sys} + \Delta v_\mathrm{r}(t). \tag{5}$$

The resulting distance is

$$r = r_0 + \int_{t_0}^{t} v_\mathrm{r}\mathrm{d}t \; = \; r_0 + (t - t_0)v_\mathrm{sys} + \Delta r(t), \tag{6}$$

where Δr is the pulsar's distance from the common barycenter with its companion. Hence we find

$$\begin{aligned} t'_n &= t_n + \frac{r_0 + (t_n - t_0)v_\mathrm{sys} + \Delta r(t_n)}{c} \\ &\overset{\text{eq. (3)}}{=} t_0 + nP + \frac{r_0 + nPv_\mathrm{sys} + \Delta r(t_n)}{c} \\ &= t'_0 + n\underbrace{P\left(1 + \frac{v_\mathrm{sys}}{c}\right)}_{\equiv P'} + \underbrace{\frac{\Delta r(t_n) - \Delta r(t_0)}{c}}_{\equiv \Delta t'_n}. \end{aligned} \tag{7}$$

If no companion were present, we would expect the pulses to arrive at times $t'_0 + nP'$, i. e. with constant intervals P'. However the radial displacement (Δr) causes a variation in light travel times, and hence, the pulse timings from those expected, $\Delta t'_\mathrm{n}$. The maximum (semi-)amplitudes of radial displacement and timing variations are related to the semi-major axis of the pulsar's orbit, a_pulsar, via:

$$\Delta t'_\mathrm{max} = \frac{\Delta r_\mathrm{max}}{c} = \frac{a_\mathrm{pulsar}\sin i}{c}, \tag{8}$$

where i is the inclination of the pulsar–companion orbit relative to the plane of the sky. Using Kepler's third law,

$$P_\mathrm{orb} = 2\pi\sqrt{\frac{a^3}{G\mathcal{M}}} \tag{9}$$

with $\mathcal{M} = \mathcal{M}_\mathrm{pulsar} + \mathcal{M}_\mathrm{comp}$, and the definition of the barycenter,

$$a = a_\mathrm{pulsar} + a_\mathrm{comp} = a_\mathrm{pulsar}\left(1 + \frac{\mathcal{M}_\mathrm{pulsar}}{\mathcal{M}_\mathrm{comp}}\right), \tag{10}$$

we obtain

$$P_\mathrm{orb} = 2\pi\sqrt{\frac{a^3_\mathrm{pulsar}\left(1 + \frac{\mathcal{M}_\mathrm{pulsar}}{\mathcal{M}_\mathrm{comp}}\right)^3}{G(\mathcal{M}_\mathrm{pulsar} + \mathcal{M}_\mathrm{comp})}} \approx 2\pi\sqrt{\frac{a^3_\mathrm{pulsar}\mathcal{M}^2_\mathrm{pulsar}}{G\mathcal{M}^3_\mathrm{comp}}} \overset{\text{eq. (8)}}{=} 2\pi\sqrt{\frac{\left(c\,\Delta t'_\mathrm{max}\right)^3\mathcal{M}^2_\mathrm{pulsar}}{G\left(\mathcal{M}_\mathrm{comp}\sin i\right)^3}}, \tag{11}$$

and after solving for the minimum mass,

$$\mathcal{M}_\mathrm{comp}\sin i = c\,\Delta t_\mathrm{max}\sqrt[3]{\frac{1}{G}\left(\frac{2\pi\mathcal{M}_\mathrm{pulsar}}{P_\mathrm{orb}}\right)^2}. \tag{12}$$

Assuming $\mathcal{M}_\mathrm{pulsar} = 1.4\,\mathcal{M}_\odot$, $P_\mathrm{orb} = 1$ yr, and $\Delta t'_\mathrm{max} = 1$ ms, we find

$$\mathcal{M}_\mathrm{comp}\sin i = 5 \times 10^{24}\ \mathrm{kg} \approx 0.8\,\mathcal{M}_\mathrm{Earth}, \tag{13}$$

i. e. we may have detected an Earth-mass planet (if the inclination is not too far away from 90°).

Extra info: the difference between emitted and observed pulse periods P and P', respectively, is due to the simple, "acoustic" doppler effect.

Problem 10.3 *(2 points)*

The gas drag force (as usual, in the Epstein regime):

$$F_{gas} = \frac{4}{3}\rho c_s \sigma v$$

where $\sigma = \pi s^2$ and v is the relative velocity of the planetesimal with respect to gas:

$$v = \eta v_K, \qquad \eta \approx \frac{c_s^2}{v_K^2}$$

The velocities c_s and v_K were already calculated in Problem 9:
$c_s \approx 2\text{kms}^{-1}$ and $v_K \approx 30\text{kms}^{-1}$, so that $\eta \approx (2/30)^2 \approx 1/200$.
Substituting other numerical values (in CGS) there results

$$F_{gas} \approx \frac{4}{3} \cdot 10^{-9} \cdot 2 \cdot 10^5 \cdot (3s^2) \cdot 30 \cdot 10^5 \cdot \left(\frac{1}{200}\right) \approx 8 \cdot 10^{-4} s^2 \cdot 1.5 \cdot 10^4 \approx 10s^2.$$

The mutual gravitational force acting upon two planetesimals with radius s "at contact" is

$$F_{gr} = \frac{Gm^2}{(2s)^2} = \frac{G}{(2s)^2}\left(\frac{4}{3}\pi\rho_{plan}s^3\right)^2 \approx \frac{G}{4s^2}\left(4\rho_{plan}s^3\right)^2 \approx 4G\rho_{plan}^2 s^4$$

or, numerically, assuming $\rho_{plan} \approx 2$,

$$F_{gr} \approx 4 \cdot 7 \cdot 10^{-8} \cdot 4s^4 \approx 10^{-6}s^4.$$

Equating F_{gas} and F_{gr} leads to

$$10s^2 = 10^{-6}s^4$$

or

$$s = 3 \cdot 10^3\text{cm} = 30m.$$

Therefore, gravity seems to be important already at sizes $\ll$ 1km, but: gas drag acts permanently, whereas mutual gravity only during (short-lasting) close encounters.

Problem 10.4 *(1 point)*

Rebounds are possible if typical velocity of fragments v is less than escape velocity of the debris cloud emerged after the collision of two planetesimals of radius s. Roughly,

$$v < v_{esc} \sim \sqrt{\frac{2Gm}{s}} \sim \sqrt{\frac{2G(4/3)\pi\rho_{plan}s^3}{s}} \sim \sqrt{8G\rho_{plan}}s$$

whence

$$s > \frac{v}{\sqrt{8G\rho_{plan}}} \sim \frac{10^3}{\sqrt{8 \cdot 7 \cdot 10^{-8} \cdot 2}} \sim \frac{10^3}{\sqrt{10^{-6}}} \sim 10^6\text{cm} \sim 10\text{km}.$$