The Solar System – Exercises

Possible solutions to Problem Set 4

Problem 4.1

The temperature of a blackbody, T, is related to the flux density emitted from its surface, F^+ via

$$F^+ = \sigma T^4, \tag{1}$$

where σ is the Stefan–Boltzmann constant and the resulting dimension is power per surface area. Assuming that this flux density is matched by the heat flow from Earth's interior, $q = 90$ mW, we obtain

$$q = \sigma T^4 \qquad \text{and} \qquad T = \left(\frac{q}{\sigma}\right)^{1/4} = 35 \text{ K}. \tag{2}$$

Earth's surface would be much colder than it is today, despite the fact that it features hot springs and active volcanoes, with giant magma chambers underneath.

Problem 4.2

The given thermal conductivity $\kappa = 3\,\mathrm{W\,m^{-1}\,K^{-1}}$ and the heat flux density $q = 90$ mW m^{-2} determine the temperature gradient $\mathrm{d}T/\mathrm{d}z$ at depth z:

$$q = \kappa \frac{\mathrm{d}T}{\mathrm{d}z} \quad \text{or} \quad \frac{\mathrm{d}T}{\mathrm{d}z} = \frac{q}{\kappa}. \tag{3}$$

If κ and q are constant, a simple relation results:

$$\frac{\Delta T}{z} = \frac{q}{\kappa}. \tag{4}$$

From a known surface temperature $T_0 = T(z = 0)$, the temperature at depth results:

$$T(z) = T(0) + \Delta T = T(0) + z\frac{q}{\kappa}. \tag{5}$$

For Earth the mean surface temperature is $T(0) \approx 288$ K.
A rough estimate could be obtained from a blackbody analysis of incident and emitted radiation flux density:

$$F_{\text{in}} = F_{\text{out}} \tag{6}$$

$$\frac{L_\odot \pi R^2}{4\pi r^2}(1 - A) = 4\pi R^2 \sigma T(0)^4. \tag{7}$$

The left-hand side of that equation depends on the solar luminosity, the distance r to Earth, Earth's albedo A and radius R. The right-hand side depends on the Stefan–Boltzmann constant σ, the equilibrium temperature and again the radius. We obtain

$$T(0) = \frac{1}{2}\sqrt[4]{\frac{(1 - A)L_\odot}{\pi \sigma r^2}}, \tag{8}$$

which results in $T(0) = 278$ K for $A = 0$.
The depth where a temperature $T(z) = 1200$ K is reached is then given by

$$z(T) = \frac{\kappa \Delta T}{q} = \frac{\kappa \cdot (T - T(0))}{q} \approx 30 \text{ km}. \tag{9}$$

For other planets, the surface temperature can be different: drastically higher in the case of Venus, which has a dense atmosphere with a strong greenhouse effect. On the other hand, Mars supposedly has a much lower heat flux coming from its (smaller) interior and its crust is therefore expected to reach those temperatures only at much greater depth. The heat conductivity brings additional uncertainties into the calculations as it may be different. The current Mars mission InSight carried a heat probe with drilling capabilities with it, but the actual drilling could not reach the depths originally intended and could not bring as many insights into Mars' surface layer as hoped for.

Problem 4.3

When a cloud of matter collapses under its own gravity to form a compact object, gravitional binding energy is released and transformed into heat. The gravitional binding energy of a ball of uniform density is given by

$$U = \frac{3}{5}\frac{GM^2}{R} = \frac{16}{15}\pi^2 G\rho^2 R^5, \tag{10}$$

where M, R, and ρ are the ball's mass, radius, and bulk density, respectively. For Earth, we obtain

$$U_\oplus \approx 2.23 \times 10^{32}\ \text{J}. \tag{11}$$

That equation can be derived from an integral, in which we add spherical shells of mass dM and radius r on top of an already existing ball of mass $M_{<r} = 4\pi\rho r^3/3$:

$$U = \int_0^M \frac{GM_{<r}\mathrm{d}M}{r} = \int_0^R \frac{GM_{<r}4\pi\rho r^2\mathrm{d}r}{r} = \int_0^R \frac{16\pi^2 G\rho^2 r^4\mathrm{d}r}{3} = \left.\frac{16\pi^2 G\rho^2 r^5}{15}\right|_0^R = \frac{16\pi^2 G\rho^2 R^5}{15}. \tag{12}$$

The energy per surface area is

$$\frac{U}{A} = \frac{U}{4\pi R^2} = \frac{4\pi G\rho^2 R^3}{15} = \frac{GM\rho}{5}, \tag{13}$$

which results in

$$\frac{U_\oplus}{A_\oplus} \approx 4 \times 10^{17}\ \text{J/m}^2. \tag{14}$$

When assuming a constant release rate over 4.5 billion years, we obtain

$$\frac{U_\oplus/A_\oplus}{4.5\ \text{Gyr}} \approx 3\ \text{W/m}^2, \tag{15}$$

which equals roughly 30 times the current heat flow from Earth's interior.

Extra info: we could also deduce a shrink rate $\dot{R}$ that releases gravitational binding energy at a rate that is sufficient to sustain today's surface heat flux density $q = 90$ mW/m^2. Balancing these powers results in

$$4\pi q R^2 = \dot{U} = \frac{\mathrm{d}U}{\mathrm{d}R}\dot{R} = -\frac{3}{5}\frac{GM^2}{R^2}\dot{R}. \tag{16}$$

We find that a very small shrink rate would suffice:

$$\dot{R} = -\frac{20\pi q R^4}{3GM^2} = -1.3 \times 10^{-12}\ \text{m/s} \approx -40\ \mu\text{m/yr}. \tag{17}$$

Bonus problem 4.4

In the first step, we derive the general shape of the radial temperature distribution, $T(r)$. An asteroid's interior temperature distribution can be described with the heat equation discussed in the lecture on planetary interiors:

$$c_p\rho\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\kappa r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right) + H, \tag{18}$$

where c_p is the specific heat capacity at constant pressure, ρ the mass density, T the temperature, t the time, r the distance from the body's center, κ the thermal conductivity, and H the heat source density. The term $\kappa\,\mathrm{d}T/\mathrm{d}r$ corresponds to the local heat flux density. We can save a lot of work when we assume an equilibrium, where $\mathrm{d}T/\mathrm{d}t = 0$. A constant κ (a homogenous body, that is) allows for further simplification:

$$0 = \frac{\kappa}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right) + H = \frac{\kappa}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + H, \tag{19}$$

where we substituted $f \equiv r^2\mathrm{d}T/\mathrm{d}r$. Assuming further that $H = \text{const}$ and separating the variables, we can integrate once:

$$\begin{aligned} -\frac{Hr^2\mathrm{d}r}{\kappa} &= \mathrm{d}f \\ -\frac{Hr^3}{3\kappa} + C_1 &= f, \end{aligned} \tag{20}$$

which introduces a constant C_1. Now we know the qualitative behaviour of $\mathrm{d}T/\mathrm{d}r$. A second integration is required to obtain $T(r)$:

$$\begin{aligned} -\frac{Hr^3}{3\kappa} + C_1 &= r^2\frac{\mathrm{d}T}{\mathrm{d}r}, \\ \left(-\frac{Hr}{3\kappa} + \frac{C_1}{r^2}\right)\mathrm{d}r &= \mathrm{d}T, \\ -\frac{Hr^2}{6\kappa} - \frac{C_1}{r} + C_2 &= T, \end{aligned} \tag{21}$$

with a second constant C_2.

The two constants can be determined from the boundary conditions. For $r \to 0$, eq. (21) approaches a singularity. This corresponds to a particular solution, where the temperature reaches infinitely high values at the center. That solution is of no practical use (for an equilibrium situation), which is why we can safely assume $C_1 = 0$. Then, the second constant is equal to the central temperature: $C_2 = T|_{r=0} = T_0$. Finally, the temperature profile is given by

$$T(r) = T_0 - \frac{Hr^2}{6\kappa}, \tag{22}$$

and the central temperature by

$$T_0 = T(R) + \frac{HR^2}{6\kappa}. \tag{23}$$

The radius that corresponds to a give T_0 is

$$R = \sqrt{\frac{6\kappa[T_0 - T(R)]}{H}} \approx \sqrt{\frac{6\kappa T_0}{H}}, \tag{24}$$

where we assumed $T_0 \gg T(R)$. Alternatively, we could assume a specific surface temperature $T(R)$ as determined by the distance to the Sun an the albedo of the surface. For Earth, the mean surface temperature is ≈ 280 K (and rising ...).

In the second step, we need to estimate the heat source density H. The required density results from the number of radioactive decays per time and volume, $\dot{n}_{40}$, which in turn is determined by the number of unstable ^{40}K atoms per volume, n_{40}, and their lifetime τ_{40}:

$$\dot{n}_{40} = n_{40}/\tau_{40}. \tag{25}$$

The lifetime is related to the halflife as

$$\tau_{40} = t_{1/2} \ln 2. \tag{26}$$

We can further assume that the ^{40}K atoms constitute a fraction $x = 0.15\,\% = 1.5 \times 10^{-3}$ of all K atoms,

$$n_{40} = x n_{\mathrm{K}}, \tag{27}$$

which contribute a fraction $y = 0.02\,\%$ to the the total mass density ρ,

$$\mu_{\mathrm{K}} u n_{\mathrm{K}} = y\rho, \tag{28}$$

where $\mu_{\mathrm{K}} = 39$ multiplied by the atomic mass unit $u = 1.67 \times 10^{-27}$ kg result in the atomic mass of potassium (which is dominated by ^{39}K). Finally, the product of volumetric decay rate and energy released per decay ($\Delta E = 1.3$ MeV $= 2.1 \times 10^{-13}$ J) is the power density

$$H = \dot{n}_{40}\Delta E = \frac{n_{40}\Delta E}{\tau_{40}} = \frac{x n_{\mathrm{K}} \Delta E \ln 2}{t_{1/2}} = \frac{x y \rho \Delta E \ln 2}{t_{1/2}\mu_{\mathrm{K}} u} = 34\ \frac{\mathrm{nW}}{\mathrm{m}^3}, \tag{29}$$

assuming $\rho = 2000$ kg/m^{-3} and $t_{1/2} = 1.248 \times 10^9$ yr. While that may seem like a small value, today's Earth (which emits $q \approx 90$ mW/m^2 through the surface) requires only an average

$$H = \frac{4\pi R^2 q}{\frac{4}{3}\pi R^3} = \frac{3q}{R} \approx 40\ \mathrm{nW/m^3}, \tag{30}$$

which is a very similar value.
Inserting $H = 34$ nW/m^3, $\kappa = 2$ W m^{-1} K^{-1} and $T_0 = 1000$ K into eq. (24), we obtain

$$R \approx 600\ \mathrm{km}, \tag{31}$$

which corresponds to the size of a dwarf planet. Meaning that this decay alone would already make objects of that size hot enough in their interiors to allow for melting and differentiation. Had we taken into account other, more short-lived, isotopes, such as ^{26}Mg (which decays to ^{26}Al), the critical radius would be even smaller.
On a side note: the asteroid's total rate of heat production (i.e. its intrinsic luminosity) is

$$L = \frac{4\pi}{3} R^3 H. \tag{32}$$

The Stefan–Boltzmann law, $L = 4\pi R^2 \sigma T(R)^4$, then allows us to estimate the surface temperature that would be caused by the internal heat alone:

$$\begin{aligned} 4\pi R^2 \sigma T^4 &= \frac{4\pi}{3} R^3 H \\ \sigma T^4 &= \frac{RH}{3} \\ T &= \left(\frac{RH}{3\sigma}\right)^{1/4}. \end{aligned} \tag{33}$$

We obtain $T(R = 600\ \mathrm{km}) \approx 19\ \mathrm{K} \ll 1000$ K, confirming our above assumption that $T(R) \ll T_0$.