The Solar System – Exercises

Possible solutions to Problem Set 3

Problem 3.1

As given in the lecture, the geometric flattening ϵ_F, the rotation parameter $\xi \equiv \omega^2 R^3/GM$, and the gravitational moment J_2 are related via

$$J_2 = \frac{2}{3}\epsilon_F - \frac{1}{3}\xi, \tag{1}$$

We can estimate ξ from Jupiter's equatorial radius $R_{eq} \approx 71000$ km, its total mass $M \approx 1.9 \times 10^{27}$ kg, and its rotation period $P = 10$ hours $= 2\pi/\omega$:

$$\xi = \frac{4\pi^2}{P^2}\frac{R^3}{GM} = 0.086. \tag{2}$$

When this result is combined with $\epsilon = 0.065$, we obtain

$$J_2 = \frac{2}{3}0.065 - \frac{1}{3}0.086 \approx 0.015, \tag{3}$$

which corresponds well to reference values in the literature.
Some additional thoughts: Could a rotating and flattened planet have $J_2 = 0$? The answer is: yes! But what would $J_2 = 0$ mean in this context? $J_2 = 0$ would mean that there is no low-order deviation from a spherical gravitational field. How is this compatible with a flat planet? If, for example, most of the planet's mass is located in a small spherical core, surrounded by a tenous gaseous atmophere, that atmosphere would not contribute much to the total gravitational field in the exterior but it would still form a flattened surface under the combined action of centrifugal force and the core's gravity. So, looking at the combination of J_2 and ξ (or J_2 and ϵ_F, or ξ and ϵ_F), we can derive some information on how the mass is distributed inside the planet.
The gravimetric flattening is given by

$$\epsilon_G \equiv \frac{g_{pole} - g_{eq}}{g_{pole}} = \frac{5}{2}\xi - \epsilon_F = 2\xi - \frac{3}{2}J_2 = 0.15, \tag{4}$$

where g_{pole} and g_{eq} are the free-fall accelerations at the poles and the equator, respectively.

Problem 3.2

In hydrostatic equilibrium, without rotation, the gradient of the pressure p is given by

$$\nabla p = \mathbf{g}(\mathbf{r})\rho(\mathbf{r}). \tag{5}$$

For shallow depths (or low heights) the density ρ and the free-fall acceleration $g \approx 10$ m s^{-2} can be assumed constant, resulting in

$$\frac{dp}{dr} = -g\rho \tag{6}$$

or

$$\Delta p = g\rho\Delta r. \tag{7}$$

Letting $\Delta p = 3 \times 10^8$ Pa, we obtain

$$\Delta r = \frac{\Delta p}{\rho g} \approx \frac{3 \times 10^8 \text{ Pa}}{3000 \text{ kg m}^{-3} \cdot 10 \text{ m s}^{-2}} = 10^4 \text{ m} = 10 \text{ km}. \tag{8}$$

Below a depth of roughly 10 km rocks yield under the pressure and can be deformed plasticly. Note that this critical pressure is 2–3 orders of magnitude lower than the bulk modulus $K_0 \sim 100$ GPa, meaning that compression is insignificant in these first 10 km. We can safely assume that ρ is constant.
Vice versa, the maximum height to which styrofoam can be stacked is given by

$$\Delta r = \frac{\Delta p}{\rho g} \approx \frac{2 \times 10^5}{20\ \mathrm{kg\ m^{-3}} \cdot 10\ \mathrm{m\ s^{-2}}} = 10^3\ \mathrm{m} = 1\ \mathrm{km}. \tag{9}$$

That tower could reach a height of 1 km – but only if it does not have to support any weight other than its own.

Bonus problem 3.3

Let's have a look at the contribution of a given mass element to the central pressure – and what happens when we move that mass element inwards. From the (spherically symmetric) equation for hydrostatic equilibrium,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -g(r)\rho(r), \tag{10}$$

we can obtain the integral

$$p(r) = -\int_{r'=R}^{r} g(r')\rho(r')\mathrm{d}r', \tag{11}$$

where the free-fall acceleration is given by

$$g(r) = \frac{GM_{<r}}{r^2}, \tag{12}$$

with

$$M_{<r} = \int_{r'=0}^{r} 4\pi r'^2 \rho(r')\mathrm{d}r' = \frac{4\pi r^3 \rho_{<r}}{3}. \tag{13}$$

The contribution of a thin spherical shell (S) of thickness $x \ll r$ to the total central pressure consists of two parts. The first part is the direct pressure exerted by the shell itself

$$p_1 = \int_{r'=r+x}^{r} \frac{GM_{<r'}\rho(r')}{r'^2}\mathrm{d}r' \approx \frac{GM_{<r}}{4\pi r^4}\int_{r'=r+x}^{r} 4\pi r'^2 \rho(r')\mathrm{d}r' = \frac{GM_{<r}M_\mathrm{S}}{4\pi r^4} \left(= \frac{G\rho_{<r}M_\mathrm{S}}{3r}\right), \tag{14}$$

where $\rho_{<r}$ is the average density for all material closer than r. The second part is indirect: the additional pressure by the shells that are further out caused by the gravitational pull towards our individual shell:

$$p_2 = \int_{r'=R}^{r} \frac{GM_\mathrm{S}\rho(r')}{r'^2}\mathrm{d}r'. \tag{15}$$

The blue region depicted in Fig. 1 causes p_1, while the red region causes p_2.

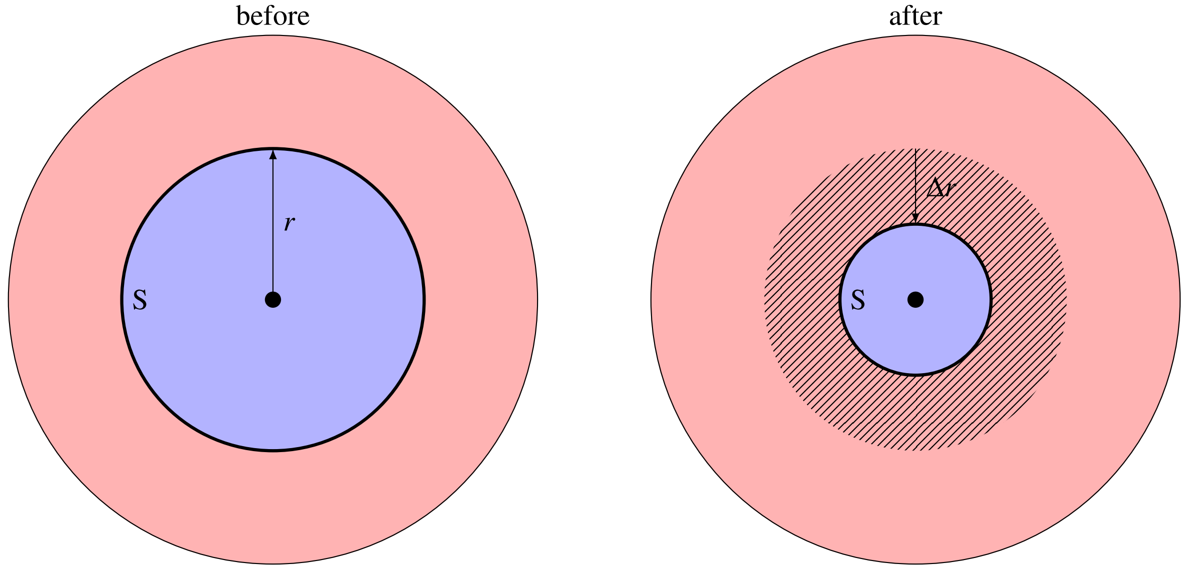

Figure 1: An inner shell (thick line) is attracted by the (blue) mass interior to it and attracts the (red) mass exterior to it. When the shell is "teleported" inwards, the (hatched) region is passed and does no longer pull the shell inwards, but instead gets pulled inward by the shell, the effect on the central pressure being the same.

If we now check what happens when we move that individual shell (without moving or squeezing the others) by a short distance Δr, both parts change. For the first part we find

$$\Delta p_1 = \Delta r \frac{\mathrm{d}p_1}{\mathrm{d}r} = \Delta r \left(\frac{\partial p_1}{\partial r} + \frac{\partial p_1}{\partial M_{<r}} \frac{\mathrm{d}M_{<r}}{\mathrm{d}r} \right) = \Delta r \left(-\frac{4 p_1}{r} + \frac{p_1}{M_{<r}} 4\pi\rho r^2 \right) = \frac{p_1 \Delta r}{r} \left(-4 + 3\frac{\rho}{\rho_{<r}} \right), \tag{16}$$

The first term (-4) is caused by the fact that the attraction increases when the distance is reduced. The second term ($+3\rho/\rho_{<r}$) modifies the pressure because the attracting inner mass $M_{<r}$ changes when the shell's radius r changes.
For the second part, the contribution by the outer shells, we find

$$\Delta p_2 = \Delta r \frac{\mathrm{d}p_2}{\mathrm{d}r} = -\Delta r \frac{G\rho M_\mathrm{S}}{r^2} = -3\frac{p_1 \Delta r}{r} \frac{\rho}{\rho_{<r}}. \tag{17}$$

This indirect part compensates the second term in Δp_1: if the individual shell is moved inwards, it will feel less attraction from the (hatched in Fig. 1) shells that it has passed on its way. But this reduction is compensated because it will now pull these other shells towards the center, increasing their contribution to the central pressure. We obtain a total change in central pressure,

$$\Delta p_1 + \Delta p_2 = -4 p_1 \frac{\Delta r}{r} = -\frac{4G\rho_{<r} M_\mathrm{S}}{3r^2} \Delta r. \tag{18}$$

Moving mass closer to the center, we have $\Delta r < 0$, and hence, $\Delta p_1 + \Delta p_2 > 0$: the central pressure increases.

Problem 3.4

In ground mode, the mode of lowest frequency, the wavelength corresponds to the distance of Earth's radius back and forth (to the center and back out):

$$\lambda \sim 2R_\oplus. \tag{19}$$

Given a typical P-wave propagation speed $v_\mathrm{P} \approx 10$ km/s, the period is

$$T = \frac{1}{f} = \frac{\lambda}{v_\mathrm{P}} = \frac{2R_\oplus}{v_\mathrm{P}} \approx \frac{1.28 \times 10^4 \text{ km}}{10 \text{ km/s}} = 1280 \text{ s} \approx 21 \text{ min}. \tag{20}$$

For spherically symmetric pulsations (contraction/expansion) of Earth as a whole, the longest measured period is 20 min, which is in good agreement. But the slowest actually measured mode has $T \approx 54$ min. That mode is more related to S waves (which propagate at lower speeds) and characterized by deformations similar to the periodic quenching and stretching of a football right after the contact with the foot. [Numbers from: T. G. Masters & R. Widmer: "Free Oscillations: Frequencies and Attenuations". In: "Global Earth Physics: A Handbook of Physical Constants", T. J. Ahrens (Hrsg.), AGU reference shelf, American Geophysical Union, Washington, DC, 1995]