The Solar System – Exercises

Possible solutions to Problem Set 2

8 Nov 2022

Bonus problem 2.1

The tower has linear dimensions on the order of ~ 100 m, which corresponds to 1 part in 4×10^5, considering Earth's circumference of 4×10^7 m. The gravitational potential would therefore need to be developed to terms that have periods on that length scale and hence as many as hundreds of thousands of nodes or order $n \gtrsim 4 \times 10^5$.

Bonus problem 2.2

We will address the problem in two ways. First, we use the well-known solution of the two-body problem that we have at hand. Relative to the asteroid it is passing, the spacecraft is on an unbound, hyperbolic trajectory (see Fig. 1). On such a trajectory, the distance to the asteroid depends on semi-major axis a, orbital eccentricity e and true anomaly θ:

$$r(\theta) = \frac{a \cdot (1 - e^2)}{1 + e \cos\theta}. \tag{1}$$

The distance is minimal for $\theta = 0$:

$$q \equiv r(\theta = 0) = a \cdot (1 - e). \tag{2}$$

The spacecraft approaches from and vanishes to infinity at specific angles $\theta_\pm$:

$$\cos\theta = -1/e \quad \text{bzw.} \quad \theta_\pm = \pm \arccos(-1/e). \tag{3}$$

If the spacecraft were passing along a straight line, the two angles would differ by 180°. The gravitational deflection by an angle α can then be expressed as

$$\theta_+ - \theta_- = 180^\circ + \alpha \quad \text{bzw.} \quad \theta_+ = 90^\circ + \alpha/2. \tag{4}$$

and (with eq. 3)

$$\sin(\alpha/2) = 1/e \quad \text{bzw.} \quad e = 1/\sin(\alpha/2) \approx 2/\alpha. \tag{5}$$

For any trajectory around it, the asteroids mass M_{ast} and semi-major axis a are related to the total (specific) energy as given by the energy constant h,

$$\frac{h}{2} = \frac{v^2}{2} - \frac{\mu}{r}, \tag{6}$$

where h

$$h = -\frac{\mu}{a} \quad \text{and} \quad \mu = G(M_{\text{ast}} + m_{\text{craft}}) \approx GM_{\text{ast}}. \tag{7}$$

At a great distance ($r \to \infty$), where the asteroid's gravitational attraction is insignificant, the unbound spacecraft travels at relative velocity $v_\infty = 10$ km/s. We obtain the semi-major axis,

$$-\frac{\mu}{2a} = \frac{h}{2} \approx \frac{v_\infty^2}{2}. \tag{8}$$

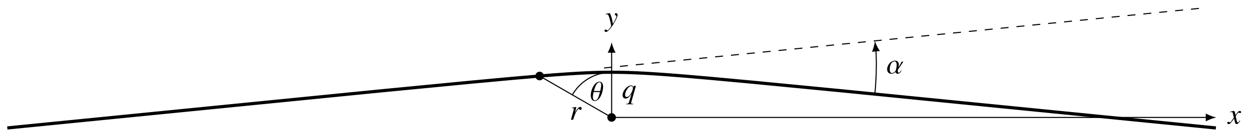

Figure 1: A spacecraft passes an asteroid (centered on the origin) on a hyperbolic trajectory.

Equation (2) leads to $a = q/(1-e)$, and hence,

$$\mu = -av^2 = \frac{qv^2}{e-1} \approx \frac{qv^2}{e} \approx \frac{\alpha q v^2}{2}. \tag{9}$$

Assuming $\alpha = 1'' = \frac{2\pi}{360\cdot 60\cdot 60}$ rad $\approx \frac{1}{2\times 10^5}$ rad, $q = 10^8$ m and $v = 10^4$ m/s we are left with

$$\mu \approx \frac{10^{16}}{2\times 10^5}\ \mathrm{m^3\ s^{-2}}, \tag{10}$$

and finally,

$$M_{\text{Asteroid}} = \frac{\mu_{\text{Asteroid}}}{G} \approx \frac{5\times 10^{10}}{7\times 10^{-11}}\ \mathrm{kg} \approx 10^{21}\ \mathrm{kg}. \tag{11}$$

The largest main-belt asteroid, (1) Ceres, has a very similar mass.
If we do not want to use the formalisms of classical celestial mechanics, we can take a different route. The deflection angle being so small ($1''$), the motion (parallel to the x axis) can be approximated as linear and with constant velocity. The asteroid exerts just a slight perpendicular force that results in a small perpendicular velocity component at the end of the fly-by. The absolute acceleration is given by

$$g = \frac{GM}{r^2}, \tag{12}$$

while the perpendicular component is

$$g_\perp \approx g_y = \frac{GM}{r^2}\cos\theta \approx \frac{GM}{r^2}\frac{y}{r}. \tag{13}$$

The accelerations along the x axis before and after the fly-by cancel out. The current distance is

$$r^2 = x^2 + y^2. \tag{14}$$

The perpendicular velocity component is then

$$\dot{y}(t) = \int\limits_{t=-\infty}^{\infty} g_y \mathrm{d}t. \tag{15}$$

Assuming a constant velocity $\dot{x} = v = \text{const}$, we find

$$\begin{aligned} \mathrm{d}x &= v\mathrm{d}t, & (16)\\ \dot{y}(t) &= \int\limits_{x=-\infty}^{\infty} \frac{g_y}{v}\,\mathrm{d}x = \int\limits_{x=-\infty}^{\infty} \frac{GM}{vr^2}\frac{y}{r}\,\mathrm{d}x & (17)\\ &= \frac{GM}{v}\int\limits_{x=-\infty}^{\infty} \frac{y\,\mathrm{d}x}{(x^2+y^2)^{3/2}}, & (18) \end{aligned}$$

and given the smallness of the deflection angle,

$$y(x) \approx \text{const}. \tag{19}$$

The integral equation could be solved directly with tabulated integrals or with a computer algebra system (such as Mathematica). Alternatively, we can do with pen and paper, reformulating first:

$$\dot{y}(t) = \frac{2GM}{v}\int\limits_{x=0}^{\infty} \frac{y\,\mathrm{d}x}{(x^2+y^2)^{3/2}} = \frac{2GM}{v}\int\limits_{x=0}^{\infty} \frac{y\,\mathrm{d}x}{x^3\left[1+\left(\frac{y}{x}\right)^2\right]^{3/2}}. \tag{20}$$

Substituting

$$\xi \equiv y/x \quad \text{and} \quad \mathrm{d}\xi = -\frac{y}{x^2}\,\mathrm{d}x, \tag{21}$$

we obtain

$$\dot{y}(t) = -\frac{2GM}{vy} \int\limits_{\xi=\infty}^{0} \frac{\xi \, \mathrm{d}\xi}{(1+\xi^2)^{3/2}}. \tag{22}$$

The enumerator is now the derivative of the discriminant in the denominator (joined by an additional factor of 2 that will be compensated by the outer derivative of the square root). This simplifies the integration drastically:

$$\dot{y}(t) = \frac{2GM}{vy} \frac{1}{(1+\xi^2)^{1/2}}\bigg|_{\xi=\infty}^{0} = \frac{2GM}{vy}. \tag{23}$$

The defelection angle is then given by

$$\alpha \approx \tan\alpha = \frac{\dot{y}}{v} = 2GMv^2y. \tag{24}$$

With the minimum distance $y = q$, we obtain

$$\alpha \approx \tan\alpha = \frac{\dot{y}}{v} = \frac{2GM}{qv^2} \qquad \text{or (again)} \qquad \underline{\underline{M = \frac{\alpha q v^2}{2G}}}. \tag{25}$$

Problem 2.3

The angular precession frequency (of the longitude of the ascending node, Ω) caused by the flattening is given by

$$\omega_\Omega = -\frac{3}{2} J_2 \omega \cos i \left[\frac{R_{\mathrm{eq}}}{a(1-e^2)}\right]^2 = -3\pi \frac{J_2}{P} \cos i \left[\frac{R_{\mathrm{eq}}}{a(1-e^2)}\right]^2, \tag{26}$$

where ω is the satellite's orbital frequency (also called mean motion), P its orbital period, i its orbital inclination, a and e its orbital semi-major axis and eccentricity, respectively. The central body is characterized by the combination of its gravitational moment J_2 and its equator radius R_{eq}. For the combination of Moon ($i \approx 29°$, $a = 384\,400$ m, $e \approx 0$, $P = 27$ days) and Earth ($J_2 \approx 0.0011$, $R_{\mathrm{eq}} \approx 6400$ km) we obtain

$$\omega_\Omega \approx 1.1 \times 10^{-12}/\mathrm{s} = \frac{2\pi}{180\,000\ \mathrm{yr}}. \tag{27}$$

That precession period of 180 000 years exceeds the Moon's actual precession period (of 18.6 years) by roughly four orders of magnitude, indicating that effects other than Earth's slight flatness dominate. The perturbations by the Sun and the other planets (most notably Jupiter) are more important, which is why the Moon's orbit is actually precessing relative to the ecliptic plane, *not* Earth's equatorial plane.