The Solar System – Exercises

Possible solutions to Problem Set 1

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Problem 1.1

The gravitational force that acts on the pendulum has two components: Earth pulls (ideally straight) downward, while the belltower (and the hill) pull horizontally. The tangent of the pendulum's angle is hence given by the ratio of these two force components:

$$\tan\alpha = \frac{F_{\rm tower}}{F_\oplus}. \tag{1}$$

(For small angles we have $\tan\alpha \approx \alpha$.) Both components can be calculate via Newton's law:

$$F_{\rm tower} = \frac{GM_{\rm tower}m_{\rm pendulum}}{d^2}, \tag{2}$$

$$F_\oplus = \frac{GM_\oplus m_{\rm pendulum}}{R_\oplus^2}, \tag{3}$$

where d is the distance between pendulum and tower. The angle is thus given by

$$\alpha \approx \frac{M_{\rm tower}}{M_\oplus}\left(\frac{R_\oplus}{d}\right)^2. \tag{4}$$

The mass of the pendulum cancels out in the ratio, but we still need estimates for d and $M_{\rm tower}$. The distance can be estimated with a map: $d \approx 250$ m. The mass estimate is more uncertain. Based on a diameter of 35 m and a height of 130 m, the volume is very roughly given by $V_{\rm tower} \approx \frac{\pi}{4} \times 35^2 \times 130\ {\rm m}^3 \approx 10^5\ {\rm m}^3$. Given that most of that volume is filled with air, a density $\rho_{\rm tower} \sim 100\ {\rm kg/m}^3$ seems appropriate, from which we obtain $M_{\rm tower} \approx 10^7$ kg and

$$\alpha \approx \frac{10^7}{6\times 10^{24}}\left(\frac{6\times 10^6}{250}\right)^2 \approx 10^{-9}\ \text{(rad)} = 0.2\ \text{mas (milliarcseconds)}, \tag{5}$$

which is an angle that is too small to be measured even by the biggest optical telescopes.
For the Hausberg, we could assume a typical density $\rho_{\rm hill} \approx 2700\ {\rm kg/m}^3$ (corresponding to average rock), a volume $V_{\rm hill}$ = height × width × length $\approx 200\ {\rm m} \times 500\ {\rm m} \times 1000\ {\rm m} = 10^8\ {\rm m}^3$, and a distance $d_{\rm hill} = 2$ km. The resulting angle is then

$$\alpha \approx 5\times 10^{-7} = 0.1''\ \text{(arcseconds)}, \tag{6}$$

which is much better, but still not easy. The Schiehallion experiment worked better because the mountain was more massive and the measurement were done directly on the mountain slopes. That way, the difference between the two measurement on either sides was $\pm 6''$.

Problem 1.2

From the maximum apparent separation, $\beta = 10'$, the true separation (and hence the orbital semi-major axis) can be obtained::

$$a_{\rm C} = \beta a_{\rm J} \approx 2\pi\frac{10'}{360^\circ}4.2a_\oplus = 2\pi\frac{10}{360\cdot 60}4.2a_\oplus = 0.012a_\oplus. \tag{7}$$

The orbital period of 17 days corresponds to 0.046 years, or

$$P_{\rm C} = 0.046P_\oplus. \tag{8}$$

With Newton's version of Kepler's 3rd law, $P^2 \propto a^3 M^{-1}$, we can relate the Callisto–Jupiter system to the Earth–Sun system:

$$\frac{M_J}{M_\odot} = \left(\frac{a_C}{a_\oplus}\right)^3 \left(\frac{P_C}{P_\oplus}\right)^{-2}. \tag{9}$$

After inserting the known quantities, the resulting mass ratio is

$$\frac{M_J}{M_\odot} = (0.012)^3 (0.046)^{-2} \approx 0.0124^{-2} \approx 10^{-3}. \tag{10}$$

Jupiter has roughly one thousandth of a Solar mass.

Bonus problem 1.3

When a planet is moving in the opposite direction for some time, it must change direction twice and hence come to an apparent halt twice. The time between these two moments is then the time span that we are looking for. At the points where the apparent motion stops briefly, the planet has only a radial velocity component relative to Earth. The relative velocity vector must therefore be parallel to the relative position vector between planet and Earth. Figure 1 shows this setting in the non-rotating Solar reference frame.
For a pair or parallel vectors, the cross product vanishes

$$(\mathbf{r} - \mathbf{r}') \times (\mathbf{v} - \mathbf{v}') = 0 \tag{11}$$

$$\mathbf{r} \times \mathbf{v} - \mathbf{r}' \times \mathbf{v} - \mathbf{r} \times \mathbf{v}' + \mathbf{r}' \times \mathbf{v}' = 0. \tag{12}$$

On a circular orbit, the velocity vectors are perpendicular to the radial position vectors. In addition, if the two orbits are coplanar, all velocities and position are in the common plan, all cross products point perpendicularly away from that plane, i. e. towards the z direction if the plane is spanned by x and y. Letting α be the angle between $\mathbf{r}$ and $\mathbf{r}'$, we obtain

$$\begin{aligned}
rv - r'v\sin(90^\circ + \alpha) - rv'\sin(90^\circ - \alpha) + r'v' &= 0 \\
rv - r'v\cos\alpha - rv'\cos\alpha + r'v' &= 0 \\
\frac{rv + r'v'}{r'v + rv'} &= \cos\alpha \\
\frac{r/r' + v'/v}{1 + (r/r')(v'/v)} &= \cos\alpha.
\end{aligned} \tag{13}$$

With $v = 2\pi r/P \propto r/r^{3/2} \propto r^{-1/2}$ (according to Kepler's 3rd law) for the orbital velocities, we find

$$\frac{r/r' + (r/r')^{1/2}}{1 + (r/r')^{3/2}} = \cos\alpha. \tag{14}$$

An alternative route to the same result begins with stating that the velocity components that are perpendicular to $\mathbf{d}$ muss be equal if they are to cancel out (see Fig. 1).

$$v\sin\beta \stackrel{!}{=} v'\sin\beta'. \tag{15}$$

To obtain the angles β and β', we can start with the trianle formed by r, r', and $d = |\mathbf{d}|$. The inner angles of this triangle are $90^\circ - \beta'$, $90^\circ + \beta$, and α. The cosine law then results in

$$d^2 = r^2 + r'^2 - 2rr'\cos\alpha, \tag{16}$$

whereas the sine law gives us

$$\frac{\sin(90^\circ - \beta')}{r} = \frac{\sin(90^\circ + \beta)}{r'} = \frac{\sin\alpha}{d}, \tag{17}$$

and hence,

$$\cos\beta' = \frac{r}{d}\sin\alpha \tag{18}$$

and

$$\cos\beta = \frac{r'}{d}\sin\alpha. \tag{19}$$

Inserting these results and $\sin^2\beta^{(\prime)} = 1 - \cos^2\beta^{(\prime)}$ back into eq. (17), we find

$$v\sqrt{1 - \frac{r'^2}{d^2}\sin^2\alpha} = v'\sqrt{1 - \frac{r^2}{d^2}\sin^2\alpha}. \tag{20}$$

Now we can multiply by d and substitute $\sin^2\alpha = 1 - \cos^2\alpha$:

$$v\sqrt{d^2 - r'^2 + r'^2\cos^2\alpha} = v'\sqrt{d^2 - r^2 + r^2\cos^2\alpha}. \tag{21}$$

In combination with eq. (18), this leads to

$$\begin{aligned} v\sqrt{r^2 - 2rr'\cos\alpha + r'^2\cos^2\alpha} &= v'\sqrt{r'^2 - 2rr'\cos\alpha + r^2\cos^2\alpha} \\ v\sqrt{(r - r'\cos\alpha)^2} &= v'\sqrt{(r' - r\cos\alpha)^2} \\ v(r - r'\cos\alpha) &= v'(r' - r\cos\alpha), \end{aligned} \tag{22}$$

and finally, again, eq. (16)
For Mars (♂), with a mean distance $r' \approx 1.5r$, the resulting angle is

$$\alpha = \pm 16^\circ. \tag{23}$$

Taking into account the synodic orbital period

$$P_{\mathrm{syn}} = (1/P_\oplus - 1/P_♂)^{-1} = 780 \text{ days}, \tag{24}$$

the time span around opposition during which Mars moves from East to West is roughly

$$t = \frac{\alpha}{2\pi}P_{\mathrm{syn}} = \pm 35 \text{ days}, \tag{25}$$

or a total of ≈ 70 days.
For very distant planets ($r' \gg r$ or $r/r' \to 0$) eq. (16) approaches

$$\cos\alpha \to 0, \quad \text{also} \quad \alpha \to \pm 90^\circ. \tag{26}$$

The very distant planet moves in the opposite direct for a full quarter of a (synodic) year before and another quarter of a (synodic) year after opposition – because it basically stands still and only moves with respect to the stellar background because Earth moves.
For very close planets ($r' \to r$), we obtain

$$\cos\alpha \to 1, \quad \text{also} \quad \alpha \to 0^\circ. \tag{27}$$

The angle approaches zero – but the time span does not because the length of the synodic year approachs infinity as the planets' orbits come closer. From $P' = P \cdot (r'/r)^{3/2}$ (Kepler's 3rd law, again ...) we obtain

$$P_{\mathrm{syn}} = \frac{P}{1 - (r/r')^{3/2}}, \tag{28}$$

and

$$t = \frac{P}{1 - (r/r')^{3/2}} \frac{\arccos\frac{r/r' + (r/r')^{1/2}}{1 + (r/r')^{3/2}}}{2\pi}. \tag{29}$$

This general solution is shown in Fig. 2. The theoretical limit for $r' \to r$ can then be approached with multiple iterations of l'Hôpital's rule:

$$t(r' \to r) = \pm\frac{P}{\sqrt{2}\cdot 3\pi} = \pm\frac{365 \text{ days}}{\sqrt{2}\cdot 3\pi} \approx \pm 27.4 \text{ days}, \tag{30}$$

corresponding to a total $2|t| \approx 55$ days.

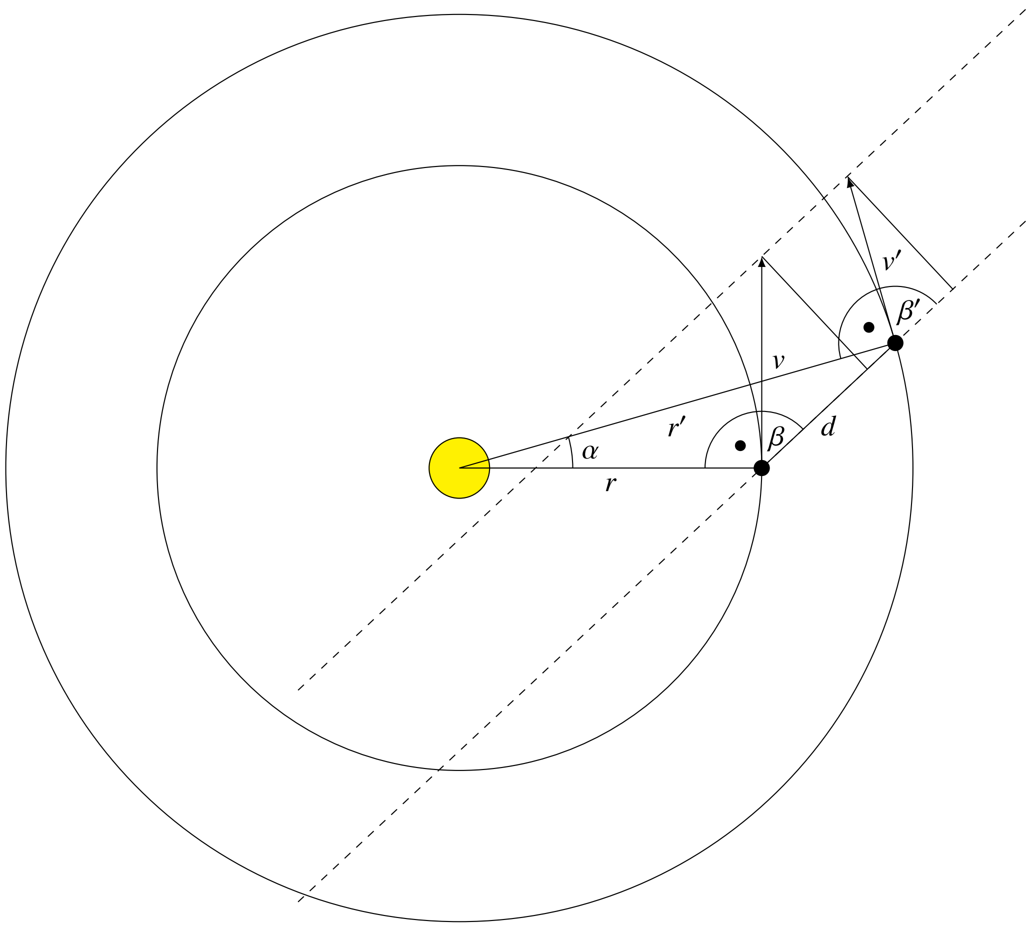

Figure 1: Positions of Sun, Earth, and Mars shortly before opposition, when Mars' apparent motion (as see from Earth) changes direction. In that moment, the relative velocity vector is parallel to the relative distance vector and the relative motion is purely radial.

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Figure 2: Time spent in East–Western motion as a function of heliocentric distance ratio r'/r (see eq. 31).