Physics of Planetary Systems — Exercises
Suggested Solutions to Set 6

Problem 6.1 *(1 point)*

Querying the catalog for HIP 66074 ($\mathscr{M}_* \approx 0.7\ \mathscr{M}_\odot$) results in an astrometric solution with a parallax

$$\varpi = (28.211 \pm 0.012)\ \text{mas}, \tag{1}$$

(corresponding to a distance $d = 1\ \text{au}/\varpi = 35.5$ pc), an apparent semi-major axis

$$\alpha = (0.213 \pm 0.033)\ \text{mas}, \tag{2}$$

and an orbital period

$$P = (297.6 \pm 2.7)\ \text{d}. \tag{3}$$

The orbital period allows us access to the sum of the masses, $\mathscr{M} \equiv \mathscr{M}_* + \mathscr{M}_\text{p}$, and semi-major axes, $a = a_* + a_\text{p}$:

$$P = 2\pi\sqrt{\frac{a^3}{G\mathscr{M}}}, \tag{4}$$

where

$$a_*\mathscr{M}_* = a_\text{p}\mathscr{M}_\text{p} = (a - a_*)\mathscr{M}_\text{p} \tag{5}$$

and hence,

$$\mathscr{M} = \mathscr{M}_* \frac{a}{a - a_*}, \tag{6}$$

resulting in

$$P = 2\pi\sqrt{\frac{a^2(a - a_*)}{G\mathscr{M}_*}} \qquad \text{or} \qquad a^3 - a^2 a_* - \underbrace{\frac{G\mathscr{M}_* P^2}{4\pi^2}}_{\equiv\xi} = 0. \tag{7}$$

This cubic equation can be solved analytically (see extra info below), but it becomes much simpler if we assume $\mathscr{M}_\text{p} \ll \mathscr{M}_*$, and hence, $a \approx a_\text{p} \gg a_*$:

$$a_\text{p} \approx \sqrt[3]{\frac{G\mathscr{M}_* P^2}{4\pi^2}} = 0.77\ \text{au}. \tag{8}$$

The stellar semi-major axis is

$$a_* = \alpha d = \alpha \frac{1\ \text{au}}{\varpi} = 0.0075\ \text{au}, \tag{9}$$

from which we obtain the planet mass

$$\mathscr{M}_\text{p} \ = \ \frac{a_*}{a_\text{p}}\mathscr{M}_* \ = \ 0.0097\,\mathscr{M}_* \ \approx \ 7.2\,\mathscr{M}_\text{Jup} \ \ll \ \mathscr{M}_*. \tag{10}$$

Extra info: the full solution to the cubic eq. (7) is

$$a = \frac{a_*}{3}\left[1 + \frac{1}{C} + C\right], \tag{11}$$

where the middle term dominates for $\mathscr{M}_* \gg \mathscr{M}_\text{p}$ because

$$C = \sqrt[3]{\frac{27\xi}{2a_*^3} + 1 - \sqrt{\left(\frac{27\xi}{2a_*^3}\right)^2 + 27\frac{\xi}{a_*^3}}} = \sqrt[3]{\frac{27\xi}{4a_*^3}\left(\sqrt{1 + \frac{4a_*^3}{27\xi}} - 1\right)^2} = \frac{a_*}{3\sqrt[3]{\xi}}\left[1 - \frac{2a_*^3}{81\xi} + \mathscr{O}(a_*^6)\right] \ll 1 \tag{12}$$

and $\sqrt[3]{\xi} \approx a_\text{p} \gg a_*$.

Problem 6.2 *(2 points)*

If you have a telescope with focal length $F = 3 \times 39$ m $= 117$ m, the plate scale will be

$$\text{plate scale} = \frac{360 \times 60 \times 60''}{2\pi F} \approx \frac{206\,265''}{F} = \frac{206\,265''}{117\,000\ \text{mm}} = 1.76''/\text{mm}. \tag{13}$$

For a pixel size of 15 µm, this corresponds to a plate scale of 0.02644″/pixel. A displacement by one hundreth of a pixel corresponds to an astrometric perturbation of 0.0002644″ (0.2644 mas).
The astrometric perturbation is

$$\theta[''] = \frac{\mathscr{M}_{\text{planet}}[\mathscr{M}_\odot]}{\mathscr{M}_*[\mathscr{M}_\odot]} \frac{a[\text{au}]}{d[\text{pc}]}, \tag{14}$$

where a is the semi-major axis and d is the distance to α Cen. This gives the planet mass

$$\mathscr{M}_{\text{planet}}[\mathscr{M}_\odot] = \theta['']\frac{\mathscr{M}_*[\mathscr{M}_\odot] d[\text{pc}]}{a[\text{au}]} = 2.64 \times 10^{-4} \frac{1 \times 1.34}{1} = 3.53 \times 10^{-4} \quad \Rightarrow \quad \mathscr{M}_{\text{planet}} = 0.37\ \mathscr{M}_{\text{Jupiter}}. \tag{15}$$

Bonus problem 6.3 *(1 extra point)*

Suppose you had a big CCD detector with 4096×4096 pixels. With a plate scale of 0.02644″ per pixel your field of view would only be 1.8′. It would be hard to find reference stars for the measurement, particularly bright ones. Another problem of course is you probably cannot use a 39-m telescope to observe such a bright star!

Problem 6.4 *(2 points)*

The incoming energy per unit time is the cross-section area of a spherical grain times the flux of stellar radiation at a distance r:

$$\pi s^2 \cdot \frac{L_*}{4\pi r^2}, \tag{16}$$

where we assume that all intercepted radiation is absorbed, i. e. a pitch black surface with Bond albedo $A = 0$. The outgoing energy per unit time is the surface area of the grain times the Stefan–Boltzmann radiation flux:

$$4\pi s^2 \cdot \sigma T_{\text{dust}}^4 \tag{17}$$

In the equilibrium both are equal:

$$\pi s^2 \cdot \frac{L_*}{4\pi r^2} = 4\pi s^2 \cdot \sigma T_{\text{dust}}^4 \tag{18}$$

so that

$$T_{dust} = \left(\frac{L_*}{\pi\sigma}\right)^{1/4} \frac{1}{\sqrt{r}} \tag{19}$$

Numerically, for the Sun at 1 au:

$$T_{dust} \approx \left(\frac{4 \cdot 10^{33}}{16 \cdot 3 \cdot 5.7 \cdot 10^{-5}}\right)^{1/4} \frac{1}{\sqrt{1.5 \cdot 10^{13}}} \approx \left(10^{36}\right)^{1/4} \frac{1}{\sqrt{15 \cdot 10^6}} \approx \frac{10^3}{\sqrt{15}} \approx 250\ \text{K}. \tag{20}$$

The temperature of 1500 K required for sublimation is 6 times higher.
It is reached at $1\ \text{au}/6^2 = 1/36\ \text{au} \approx 6 R_\odot$.

Problem 6.5 *(2 points)*

The sound speed was estimated in an earlier problem: $c_\mathrm{s} \sim 1$ km/s $\sim 10^3$ m/s at 1 au. The resulting gas damping constant is

$$\Gamma = \frac{\rho_\mathrm{gas}}{\rho_\mathrm{dust}} \frac{c_\mathrm{s}}{s} \sim \frac{10^{-9}\ \mathrm{g/cm^3}}{1\ \mathrm{g/cm^3}} \frac{10^3\ \mathrm{m/s}}{s} \sim \frac{10^{-6}\ \mathrm{m/s}}{s}, \tag{21}$$

and the Kepler frequency

$$\Omega_\mathrm{K} \sim \frac{2\pi}{1\ \mathrm{yr}} \sim \frac{6}{3 \cdot 10^7\ \mathrm{s}} \sim 2 \cdot 10^{-7}/\mathrm{s}. \tag{22}$$

The condition

$$\Gamma \stackrel{!}{=} 2\Omega_\mathrm{K} \tag{23}$$

becomes

$$\frac{10^{-6}\ \mathrm{m/s}}{s} \sim 2 \cdot 10^{-7}/\mathrm{s} \tag{24}$$

or

$$s \sim 5\ \mathrm{m}. \tag{25}$$

Therefore, the boundary is at meter-sized bodies.

An alternative, more general expression can be found the following way:

$$\begin{aligned}
\frac{\rho_\mathrm{gas}}{\rho_\mathrm{dust}} \frac{c_\mathrm{s}}{s} &= 2\Omega_\mathrm{K} && \text{where } \Omega_\mathrm{K} = v_\mathrm{K}/r \\
\frac{\rho_\mathrm{gas}}{\rho_\mathrm{dust}} \frac{c_\mathrm{s}}{s} &= 2\frac{v_\mathrm{K}}{r} && \text{where } \frac{c_\mathrm{s}}{v_\mathrm{K}} = \frac{h}{r} \\
\frac{\rho_\mathrm{gas}}{\rho_\mathrm{dust}} \frac{h}{2} &= s && \text{where } \Sigma_\mathrm{gas} = \rho_\mathrm{gas} h \\
\frac{1}{2} \frac{\Sigma_\mathrm{gas}}{\rho_\mathrm{dust}} &= s.
\end{aligned}$$

Assuming further that $\Sigma_\mathrm{gas} \approx 100\Sigma_\mathrm{dust}$, we obtain

$$s \approx 50 \frac{\Sigma_\mathrm{dust}}{\rho_\mathrm{dust}}, \tag{26}$$

where Σ_dust is the usual surface mass density of dust (on the order of 10 g/cm^2) and ρ_dust the bulk density (on the order of 1 g/cm^3), resulting again in $s \sim 5$ m.