Physics of Planetary Systems — Exercises
Suggested Solutions to Set 5

Problem 5.1 *(2 points)*

The transit probability is just $p = R_{\mathrm{star}}/a$. The photometric amplitude is given by $\Delta F/F = (R_{\mathrm{p}}/R_*)^2$. An expression for the transit duration was given in class:

$$\tau = 2R_{\mathrm{star}} \left[\frac{P}{2\pi G \mathscr{M}_{\mathrm{star}}}\right]^{1/3} = 1.82 \text{ hours} \times \frac{R_{\mathrm{star}}}{R_{\mathrm{sun}}} \left[\frac{P}{1 \text{ day}} \frac{\mathscr{M}_{\mathrm{sun}}}{\mathscr{M}_{\mathrm{star}}}\right]^{1/3}, \tag{1}$$

which can also be reformulated to

$$\tau = 2R_{\mathrm{star}} \left[\frac{a}{G \mathscr{M}_{\mathrm{star}}}\right]^{1/2} = 13 \text{ hours} \times \frac{R_{\mathrm{star}}}{R_{\mathrm{sun}}} \left[\frac{a}{1 \text{ au}} \frac{\mathscr{M}_{\mathrm{sun}}}{\mathscr{M}_{\mathrm{star}}}\right]^{1/2}. \tag{2}$$

The maximum RV amplitude (assuming an edge-on orbit, inclination $i = 0°$) is

$$K_{\mathrm{star}} = \mathscr{M}_{\mathrm{p}} \sqrt[3]{\frac{2\pi G}{P \mathscr{M}_{\mathrm{star}}^2}} = \frac{\mathscr{M}_{\mathrm{p}}}{\mathscr{M}_{\mathrm{star}}} \underbrace{\sqrt{\frac{G \mathscr{M}_{\mathrm{star}}}{a}}}_{v_{\mathrm{K}}} = 28.4 \text{ m/s} \times \frac{\mathscr{M}_{\mathrm{p}}[\mathscr{M}_{\mathrm{Jup}}]}{\sqrt{a[\mathrm{au}]} \mathscr{M}_{\mathrm{star}}[\mathscr{M}_{\mathrm{Sun}}]}. \tag{3}$$

These equations can be derived easily using Kepler's laws and assuming circular orbits.

(a) With $a = 0.1$ au $= 21.4\ R_{\mathrm{sun}}$ and $R_{\mathrm{star}} = R_{\mathrm{sun}}$, we obtain $p = 0.046$. Neptune has a radius that is 0.035 times that of the Sun. The photometric amplitude is thus $0.035^2 = 0.001225 = 0.12\%$. For calculating the transit duration, we first assume $R_{\mathrm{star}} \gg R_{\mathrm{planet}}$. From $\mathscr{M}_{\mathrm{star}} = \mathscr{M}_{\mathrm{sun}}$ and equation (2) we find $\tau = 4.1$ hours. Eq. (3) with $\mathscr{M}_{\mathrm{p}} = 1.02 \times 10^{29}$ g $= 0.054\ \mathscr{M}_{\mathrm{Jup}}$ and $a = 0.1$ au results in $K_{\mathrm{star}} \approx 4.8$ m/s.

(b) For a start, we need to know the radius of a K0III star, which can range from 8 to 20 solar radii. We will use an intermediate value, $R_{\mathrm{star}} = 15\ R_{\mathrm{sun}} = 0.07$ au. Thus, the transit probability for our case is $p = R_{\mathrm{star}}/a = 0.07\ \mathrm{au}/2\ \mathrm{au} = 0.035 = 3.5\,\%$. The photometric amplitude is then given by $\Delta I/I = (1/15)^2 = 0.004 = 0.4\,\%$. This looks similar to a transiting Neptune! In order to calculate the transit duration, we need to assume a stellar mass. Masses of red giants are well known and can span 1–2 $\mathscr{M}_{\mathrm{sun}}$. Let us assume a solar mass for the moment. Hence, $a = 2$ au implies an orbital period $P = 2.82$ years $= 1030$ days. The transit duration is thus $\tau = 276$ hours $= 11.5$ days, i.e. we now know it is not a transiting Neptune! Even if we assumed $R = 1\ R_{\mathrm{sun}}$, we still get a transit time of 18.4 hrs. Thus the transit duration can be used to get an estimate of how big your star is. (For the mass ($1.7\ \mathscr{M}_{\mathrm{sun}}$) and radius ($9\ R_{\mathrm{sun}}$) of the specific K0IIIb star Pollux, we obtain $p = 0.021 = 2.1\,\%$, $\Delta I/I = 0.012 = 1.2\,\%$, and $\tau = 127$ hours $= 5.3$ days.) The resulting max. RV amplitude is $K_{\mathrm{star}} \approx 16$ km/s (for $\mathscr{M}_{\mathrm{p}} = \mathscr{M}_{\mathrm{Sun}}$, $\mathscr{M}_{\mathrm{star}} = 1.7\mathscr{M}_{\mathrm{Sun}}$, $a_{\mathrm{p}} = 2$ au). Note that our assumption of $\mathscr{M}_{\mathrm{star}} \gg \mathscr{M}_{\mathrm{p}}$ does not hold for this configuration, meaning that our estimates will be pretty rough.

Problem 5.2 *(3 points)*

Possible sources of false positives:

- Grazing eclipse by a binary: can easily be distinguished with radial velocity measurements which would show an amplitude of several tens of km/s instead of hundreds of m/s.
- Transit of a main sequence star across a giant. A spectrum of the host star should reveal it is a giant. Plus the transit duration will be too long. A transit across a giant star can take many tens of hours to days.
- Eclipsing binary in background diluted by the light of a bright foreground object. This is difficult to resolve with radial velocity measurements. Probably need very high resolution imaging, or spectra in the infrared.
- Hierarchical binary, i.e. an eclipsing binary in orbit around a brighter star. High resolution imaging is needed to resolve system, or infrared measurements. Depending on the orbital period of the binary about the main star, one could see a radial velocity trend due to a binary star.

Problem 5.3 *(1 point)*

First, one can try to image those inner holes directly – and this has been done successfully for a few objects! Second, the evidence can be found in the SEDs of a large number of disks. In an SED, one can see the signatures of the dust on top of the stellar photosphere in the IR and at longer wavelengths. This dust emission comes from different disk regions. Roughly, the near-IR emission (1–5 μm) is dominated by dust close to the star around the sublimation distance. The mid-IR emission traces the inner tens of astronomical units, and the emission at longer wavelengths comes from farther out. Some disks show significant near- and far-infrared excesses relative to their stellar photospheres, but exhibit mid-infrared dips. This hints at a gap between the inner and outer regions. However, SEDs are not spatially resolved and their information must be supplemented by imaging to confirm a gap.

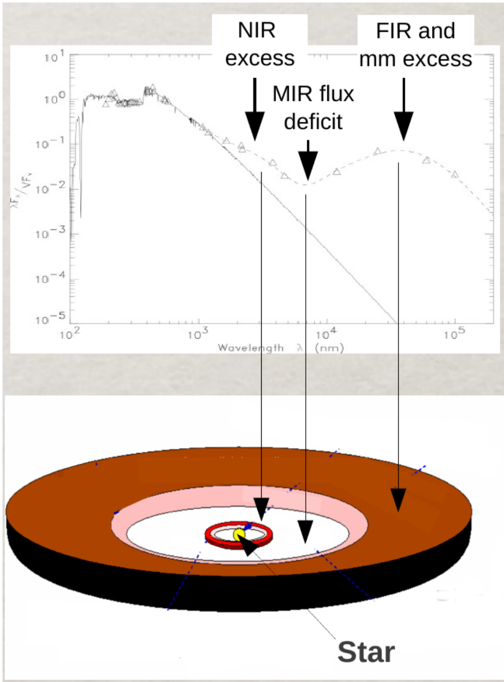

Figure 1: Schematic of a transitional disk structure (credit: A. Matter)

Bonus problem 5.4 *(1 extra point)*

The TOOMRE parameter indicates the ability of a fraction of a protoplanetary gas disk to collapse under its own gravity. The straightforward version of the parameter is based on *local* quantities (reflecting the fact that, well, the collapse is something local):

$$Q \equiv \frac{h\Omega_K^2}{\pi G \Sigma}. \tag{4}$$

Inserting $\Omega_K^2 = v_K^2/r^2 = G\mathscr{M}_*/r^3$, we find

$$Q = \frac{h}{r}\frac{\mathscr{M}_*}{\pi r^2 \Sigma}. \tag{5}$$

The surface mass density $\Sigma = \Sigma(r)$ can vary accross a disk, but if we assume an average value, such that

$$\mathscr{M}_{\text{disk}} = \pi R_{\text{disk}}^2 \Sigma, \qquad R_{\text{disk}} \sim r, \tag{6}$$

we obtain

$$Q \approx \frac{h}{r}\frac{\mathscr{M}_*}{\mathscr{M}_{\text{disk}}}, \tag{7}$$

which involves the total disk mass, a global parameter.

Problem 5.5 *(1 point)*

Giant planets can form directly, when the disk is gravitationally unstable. The Toomre instability criterion is expressed as

$$Q \equiv \frac{h}{r}\frac{\mathscr{M}_\star}{\mathscr{M}_{\rm disk}} < 2. \tag{8}$$

Denoting $\varepsilon \equiv \mathscr{M}_{\rm disk}/\mathscr{M}_\star$ and using the formula for the scale height, $h = c_s/\Omega_K$ or $h/r = c_{\rm s}/v_{\rm K}$, we rewrite the criterion as

$$\frac{c_{\rm s}}{v_{\rm K}} < 2\varepsilon. \tag{9}$$

For the sound velocity we have

$$c_{\rm s} = \sqrt{\frac{kT}{\mu m_{\rm p}}} \tag{10}$$

yielding

$$\sqrt{\frac{kT}{\mu m_{\rm p}}} < 2\varepsilon v_{\rm K} \tag{11}$$

or

$$T < 4\varepsilon^2 \frac{\mu m_{\rm p}}{k} v_{\rm K}^2 \tag{12}$$

Here, the Kepler circular velocity is given by $v_{\rm K} = \sqrt{G\mathscr{M}_\star/r}$, resulting in

$$T < 4\varepsilon^2 \frac{\mu m_{\rm p} G\mathscr{M}_\star}{kr} \propto \varepsilon^2 \mathscr{M}_\star r^{-1}. \tag{13}$$

Assuming $\mathscr{M}_\star = \mathscr{M}_\odot$, we obtain $v_{\rm K} = 30$ km/s $= 3 \cdot 10^6$ cm/s at $r = 1$ au. With $\varepsilon = 0.01$ and $\mu = 2$ (molecular hydrogen), the Toomre instability criterion is

$$T < T_{\rm T} = 4 \times 10^{-4}\ \frac{2 \times 1.7 \times 10^{-24}}{1.4 \times 10^{-16}} \times (3 \times 10^6)^2 \sim 100\ \text{K}, \tag{14}$$

which is too cold! At Saturn's distance of 10 au (meaning a much more extended disk of the same mass), the required temperature is as low as 10 K, which is absolutely unrealistic. But for $\varepsilon \sim 0.1$ instead of 0.01 the critical temperature grows by two orders of magnitude, reaching reasonable values.

Bonus: Gammie's cooling time (see lecture notes) can be translated to a critical temperature in the following way:

$$\begin{aligned} \frac{k\Sigma}{\sigma\mu m_{\rm p} T^3} \sim \tau_{\rm c} \quad &\overset{!}{\lesssim} \quad \frac{P}{2} = \pi\sqrt{\frac{r^3}{G\mathscr{M}_*}} \\ T_{\rm G} \quad &\gtrsim \quad \sqrt[3]{\frac{k\Sigma}{\pi\sigma\mu m_{\rm p}}\sqrt{\frac{G\mathscr{M}_\star}{r^3}}}. \end{aligned} \tag{15}$$

Using the same approximation as in Prob. 5.4, $\mathscr{M}_{\rm disk} \approx \Sigma\pi r^2$, we find

$$T \gtrsim T_{\rm G} = \sqrt[3]{\frac{k\varepsilon}{\pi^2 r^2 \sigma\mu m_{\rm p}}\sqrt{\frac{G\mathscr{M}_\star^3}{r^3}}} \propto \varepsilon^{1/3}\mathscr{M}_\star^{1/2} r^{-7/6}. \tag{16}$$

For the same parameters as above we obtain

$$T_{\rm G} \approx 1000\ \mathrm{K}. \tag{17}$$

In total, the temperature must be between the two critical values:

$$T_{\rm G} \lesssim T < T_{\rm T}, \tag{18}$$

which implies $T_{\rm G} < T_{\rm T}$ or

$$1 > \frac{T_{\rm G}}{T_{\rm T}} \propto \varepsilon^{-5/3} \mathscr{M}_\star^{-1/2} r^{-1/6}. \tag{19}$$

Whether the GI scenario is possible at all is most strongly determined by the mass ratio ε, with only a week dependence on distance. In the above example, where $T_{\rm G} \gg T_{\rm T}$, the scenario appears highly unlikely. Again, only for greater mass ratios (such as $\varepsilon = 0.1$) would it become possible.