Physics of Planetary Systems — Exercises
Suggested Solutions to Set 4

Problem 4.1 *(4 points)*

The provided script downloads the light curve for TOI 715 and computes a periodogram (Fig. 1), which clearly peaks at a period

$$\text{(c)} \quad P = 19.29\text{ d} = 0.0528\text{ yr}. \tag{1}$$

Given a stellar mass $\mathscr{M}_* = 0.23\mathscr{M}_\odot$ (and assuming that the companion candidate is of much lower mass), we obtain an orbital semi-major axis

$$\text{(d)} \quad a = \sqrt[3]{\frac{G\mathscr{M}_* P^2}{4\pi^2}} = 1\text{ au} \times \sqrt[3]{\mathscr{M}_*[\mathscr{M}_\odot] P[\text{yr}]^2} = 0.086\text{ au}. \tag{2}$$

The phase-folded light curve shown in Fig. 2 is then fitted with a simple box transit model. The best-fitting transit duration is

$$\text{(b)} \quad \frac{\Delta F}{F} = 0.0039 \pm 0.0002 = (3.9 \pm 0.2)\text{‰}, \tag{3}$$

which is related to the ratio of radii:

$$\frac{R_\text{p}}{R_*} = \sqrt{\frac{\Delta F}{F}} \qquad \longrightarrow \qquad R_\text{p} = R_* \sqrt{\frac{\Delta F}{F}} \approx 0.063 R_* \qquad \longrightarrow \qquad R_\text{p} \ll R_*. \tag{4}$$

From the fitted transit duration,

$$\text{(a)} \quad \tau = 0.079\text{ d}, \tag{5}$$

and the general relation (see lecture)

$$\tau = T_\text{tr} = \frac{PR_*}{a} \frac{\sqrt{(1+R_\text{p}/R_*)^2 - b^2}}{\pi} \underbrace{\frac{(1-e^2)}{1+e\sin\omega}}_{=1,\text{ for } e=0} \overset{R_\text{p}\ll R_*}{\approx} \frac{PR_*}{a}\frac{\sqrt{1-b^2}}{\pi} \overset{b\ll 1}{\approx} \frac{PR_*}{\pi a}, \tag{6}$$

we can then deduce the stellar radius,

$$\text{(e)} \quad R_* = \pi a \frac{\tau}{P} = 0.00111\text{ au} = 0.24\, R_\odot, \tag{7}$$

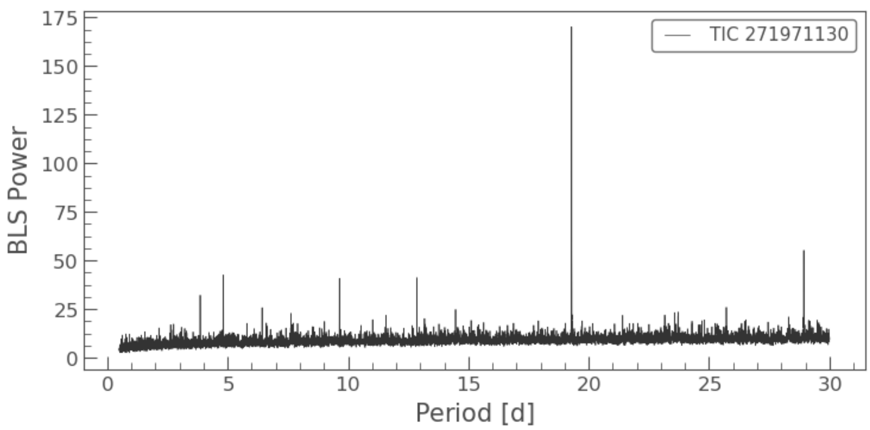

Figure 1: Periodogram for the TOI 715 light curve.

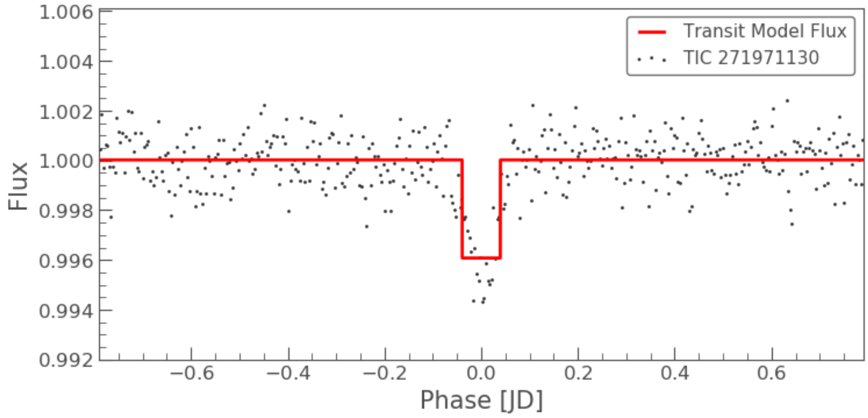

Figure 2: Light curve of TOI 715, phase-folded to a period of 19.29 d.

the transit probability,

$$\text{(f)} \quad p_{\text{tr}} = \frac{R_*}{a} = \pi \frac{\tau}{P} = 1.2\ \%. \tag{8}$$

and the radius of the planet candidate,

$$\text{(g)} \quad R_{\text{p}} = R_* \sqrt{\frac{\Delta F}{F}} \approx 0.063\ R_* = 0.015\ R_\odot \approx 1.6\ R_\oplus. \tag{9}$$

Finally, the expected RV amplitude is given by

$$K_1 = \mathscr{M}_2 \sin i \sqrt[3]{\frac{2\pi G}{P \mathscr{M}_1^2}}, \tag{10}$$

where the inclination is close enough to 90° (because it's a transit) to assume $\sin i \approx 1$. To get an estimate, we need an estimate for the companions mass. Based on its radius, we can assume that this is a "super earth", i.e. something similar to Earth in composition, just a bit larger. Assuming equal densities, we find

$$\frac{\mathscr{M}_{\text{p}}}{\mathscr{M}_\oplus} = \left(\frac{R_{\text{p}}}{R_\oplus} \right)^3 \approx 4.4, \tag{11}$$

and hence,

$$\text{(h)} \quad K_1 = 2.8\ \text{m/s}. \tag{12}$$

Problem 4.2 *(3 points)*

Assume power laws

$$c_s^2 \propto T \propto r^{-\xi} \qquad \text{and} \qquad \Sigma \propto r^{-\zeta}, \tag{13}$$

so that

$$\nu = \alpha \frac{c_s^2}{\Omega_K} \propto r^{-\xi+3/2}. \tag{14}$$

Substitute these into the formula for the radial velocity

$$v_r = -\frac{3\nu}{2r} - \frac{3}{\Sigma} \frac{\partial}{\partial r}(\Sigma \nu) = -\frac{3}{\Sigma \sqrt{r}} \frac{\partial}{\partial r} \left(\Sigma \nu \sqrt{r} \right) \tag{15}$$

to get

$$v_r \propto \frac{\nu}{r} \propto r^{-\xi+1/2} \tag{16}$$

Now, the stationary continuity equation,

$$\frac{\partial(\Sigma r v_r)}{\partial r} = 0, \tag{17}$$

requires $\Sigma r v_r = \text{const}$ or

$$r^{-\zeta} \cdot r \cdot r^{-\xi+1/2} = r^{-\zeta} \cdot r^{-\xi+3/2} = \text{const}, \tag{18}$$

whence

$$\zeta = -\xi + 3/2. \tag{19}$$

Therefore, a general solution is

$$T \propto r^{-\xi}, \qquad \nu \propto r^{-\xi+3/2}, \qquad \Sigma \propto r^{\xi-3/2}. \tag{20}$$

To be "physical", these solutions must have at least $\xi > 0$ (the farther out from the star, the colder). On the other hand, $\xi < 3/2$ is a reasonable requirement because the surface density is not expected to grow outward. These limitations are not strict though.

Note that a steepening temperature profile ($\sigma \uparrow$) results in a shallower density profile and vice versa. This can be understood by looking again at the continuity criterion. The product rv_r is roughly proportional to ν and thus to c_s^2 and T. Hence the product ΣT is conserved in the stationary case. The profiles of the two quantities must therefore compensate.

Plotting several of these solutions, for instance for $\xi = 0$, $1/2$, 1, and $3/2$ is straightforward. Hopefully you will not do that in linear scale ... log-log is the most natural scale to plot power laws.

Bonus problem 4.3 *(2 extra points)*

As shown in the lecture, the radial momentum equation for the gas disk has the form

$$-\rho\frac{v_\phi^2}{r} = -\frac{\partial(\rho c_s^2)}{\partial r} - \rho\frac{G\mathscr{M}_\star}{r^2}$$

where

$$\rho c_s^2 = n\mu m_p \cdot kT/(\mu m_p) = nkT = p \text{ (pressure)}. \tag{21}$$

Keeping our approach a bit more general, let us assume that pressure is a power law of distance from the star:

$$p = C \cdot r^{-a}. \tag{22}$$

Then, the pressure gradient term in the radial momentum equation can be written as

$$-\partial(\rho c_s^2)/\partial r = -\mathrm{d}p/\mathrm{d}r = aCr^{-a-1} = ap/r = a\rho c_s^2/r,$$

and the whole equation simplifies to

$$-\rho\frac{v_\phi^2}{r} = a\frac{\rho c_s^2}{r} - \rho\frac{v_K^2}{r}$$

or

$$v_\phi^2 = v_K^2 - ac_s^2$$

or

$$v_\phi \approx v_K(1-\eta),$$

where we define

$$\eta \equiv \frac{a}{2}\frac{c_s^2}{v_K^2}$$

for convenience.

From the equations of viscous accretion, we obtain (see Prob. 4.2)

$$T \propto r^{-\xi}, \qquad \Sigma \propto r^{\xi-3/2}.$$

When inserted into eq. (21) this results in

$$p = nkT \propto \Sigma T \propto r^{-3/2}. \tag{23}$$

Comparison of eqs. (22) and (23) shows that $a = 3/2$, and hence

$$v_\phi \approx v_K(1-\eta) \qquad \text{with} \qquad \eta = \frac{3}{4}\frac{c_s^2}{v_K^2}.$$