Physics of Planetary Systems — Exercises
Suggested Solutions to Set 2

Problem 2.1

(2 points)

In the lecture the observed radial velocity amplitude of the primary (star), $K_1 = v_{\text{obs}}$, was derived:

$$\underbrace{\frac{\mathscr{M}_2^3 \sin^3 i}{\left(\mathscr{M}_1 + \mathscr{M}_2\right)^2}}_{\equiv f} = \frac{K_1^3 P \left(1-e^2\right)^{3/2}}{2\pi G}. \tag{1}$$

where f is the mass function. Hence we have

$$v_{\text{obs}} = K_1 = \frac{\sqrt[3]{2\pi G f / P}}{\sqrt{1-e^2}}. \tag{2}$$

Assuming $\mathscr{M}_1 \gg \mathscr{M}_2$ and $e \approx 0$, we get an approximate

$$v_{\text{obs}} \approx \sqrt[3]{\frac{2\pi G \mathscr{M}_2^3 \sin^3 i}{P \mathscr{M}_1^2}} = \mathscr{M}_2 \sin i \sqrt[3]{\frac{2\pi G}{P \mathscr{M}_1^2}}. \tag{3}$$

Now we can expand with the suggested units (and replace some variables): $\mathscr{M}_2 = m_{\text{p}}$ in Jupiter masses, P in years, and $\mathscr{M}_1 = m_{\text{s}}$ in Solar masses:

$$\begin{aligned} v_{\text{obs}} &\approx \mathscr{M}_{\text{Jup}} \left(\frac{m_{\text{p}}}{\mathscr{M}_{\text{Jup}}}\right) \sin i \sqrt[3]{\frac{2\pi G / \left(\text{yr}\mathscr{M}_{\text{Sun}}^2\right)}{\left(P/\text{yr}\right)\left(m_{\text{s}}/\mathscr{M}_{\text{Sun}}\right)^2}} \\ &= \mathscr{M}_{\text{Jup}} \sqrt[3]{\frac{2\pi G}{\text{yr}\mathscr{M}_{\text{Sun}}^2}} \times \frac{m_{\text{p}}[\mathscr{M}_{\text{Jup}}] \sin i}{P[\text{yr}]^{1/3} m_{\text{s}}[\mathscr{M}_{\text{Sun}}]^{2/3}}. \end{aligned} \tag{4}$$

The pre-factor is constant and can be computed directly from its constituents. Alternatively, we can tranform it further, considering that 1 yr is Earth's orbital period:

$$1 \text{ yr} = 2\pi \sqrt{\frac{(1 \text{ au})^3}{G\mathscr{M}_{\text{Sun}}}}, \tag{5}$$

from which we obtain

$$\mathscr{M}_{\text{Jup}} \sqrt[3]{\frac{2\pi G}{\text{yr}\mathscr{M}_{\text{Sun}}^2}} = \frac{\mathscr{M}_{\text{Jup}}}{\mathscr{M}_{\text{Sun}}} \sqrt[3]{\frac{2\pi G \mathscr{M}_{\text{Sun}}}{\text{yr}}} = \frac{\mathscr{M}_{\text{Jup}}}{\mathscr{M}_{\text{Sun}}} \sqrt{\frac{G\mathscr{M}_{\text{Sun}}}{1 \text{ au}}} = \underbrace{\frac{\mathscr{M}_{\text{Jup}}}{\mathscr{M}_{\text{Sun}}}}_{1/1050} \underbrace{v_{\text{Earth}}}_{30 \text{ km/s}} \approx 28.4 \text{ m/s}, \tag{6}$$

and finally,

$$v_{\text{obs}}[\text{m/s}] \approx 28.4 \times \frac{m_{\text{p}}[\mathscr{M}_{\text{Jup}}] \sin i}{P[\text{yr}]^{1/3} m_{\text{s}}[\mathscr{M}_{\text{Sun}}]^{2/3}}. \tag{7}$$

Problem 2.2

(2 points)

Recall the mass function given in the lecture:

$$f(m) = \frac{\left(m_{\text{planet}} \sin i\right)^3}{\left(m_{\text{star}} + m_{\text{planet}}\right)^2} = \frac{PK^3 \left(1-e^2\right)^{3/2}}{2\pi G}.$$

As instructed, we have to consider identical host star mass and orbital inclination for both given cases. Let m_c and m_e (both $\ll m_\mathrm{star}$) be the masses of the circular and eccentric planets, and f_c and f_e their respective mass functions. Taking the ratio of the mass functions and now using that stellar masses and $\sin i$ are the same for both stars, we have

$$\frac{f_\mathrm{e}}{f_\mathrm{c}} = \left(\frac{m_\mathrm{e}}{m_\mathrm{c}}\right)^3 = \frac{\left(1-e^2\right)^{3/2}}{1} \quad = \quad 0.08,$$

and thus $m_\mathrm{e} = 0.43\ m_\mathrm{c}$. So, the planet in the more eccentric orbit has about 40% of the mass of the one in the circular orbit.

Bonus problem 2.3 *(1 extra point)*

The actual analytic solution of the integral is not trivial. However, to solve our initial problem, a simple comparison is possible: When comparing the disk to a homegeneous ball of the same radius and mass, we note that the mean distance between two particles in a disk, r_{12}, is smaller than that in the ball. The matter is more densely packed in the disk, the binding energy is higher. Hence, U will have a greater absolute value.

Mathematically, the gravitational binding energy of any extended object can be expressed as

$$U = -\frac{G}{2}\iint \frac{d\mathscr{M}_1 d\mathscr{M}_2}{r_{12}}$$

where both integrals cover the full object. Essentially, the result is

$$U = -\frac{GM^2}{2}\left\langle \frac{1}{r_{12}} \right\rangle.$$

The gravitational bond grows with the mean inverse distance, i. e. the bond is stronger if the mean distance is shorter.

Extra info: A more detailed analysis could start with the potential energy of a homogeneous sphere of uniform density with radius R and mass $\mathscr{M}$:

$$|U| = G\int_0^{\mathscr{M}} \frac{\mathscr{M}(<r)d\mathscr{M}(r)}{r}.$$

This integral can be understood as a piece-wise assembly of the sphere from individual shells of mass $d\mathscr{M}$. This simple approach works because the gravitational potential of a spherical shell is the same as that of a point of the same mass. Each shell of radius r and width dr adds a bit of binding energy when it is layered atop the already existing inner ball of radius r and mass $\mathscr{M}(<r)$. But $\mathscr{M}(<r) = 4/3\pi\rho r^3$ and $d\mathscr{M}(r) = 4\pi\rho r^2 dr$, which results in

$$|U| = G\int_0^R \frac{\frac{4}{3}\pi\rho r^3 \cdot 4\pi\rho r^2}{r}dr = G\frac{(4\pi\rho)^2}{3}\int_0^R r^4 dr = G\frac{(4\pi\rho)^2}{3}\frac{R^5}{5} = G\frac{(4\pi\rho R^3)^2}{15R} = G\frac{(3\mathscr{M})^2}{15R} = \frac{3}{5}\frac{G\mathscr{M}^2}{R}.$$

The potential energy of a uniform thin disk is obviously

$$-\frac{G\Sigma^2}{2}\iiiint \frac{\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2}{r_{12}}$$

where we integrate over the Cartesian plane and $\Sigma = \mathscr{M}/(\pi R_\mathrm{disk}^2)$ is the surface density. Hence,

$$U = -\frac{G\mathscr{M}^2}{2\pi^2 R_\mathrm{disk}^4}\iiiint \frac{\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2}{r_{12}}.$$

Let's get back to a more quantitative solution. In polar coordinates we have

$$U = -\frac{G\mathscr{M}^2}{2\pi^2 R_\mathrm{disk}^4}\int_0^{R_\mathrm{disk}}\int_0^{R_\mathrm{disk}}\int_0^{2\pi}\int_0^{2\pi} \frac{r_1\mathrm{d}r_1 r_2\mathrm{d}r_2\mathrm{d}\phi_1\mathrm{d}\phi_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\phi_1-\phi_2)}}.$$

With $\alpha = \phi_1 - \phi_2$ (and $|\mathrm{d}(\alpha,\phi_2)/\mathrm{d}(\phi_1,\phi_2)| = 1$) this can be simplified to

$$U = -\frac{G\mathscr{M}^2}{\pi R_{\mathrm{disk}}^4}\int_0^{R_{\mathrm{disk}}}\int_0^{R_{\mathrm{disk}}}\int_0^{2\pi}\frac{r_1\mathrm{d}r_1 r_2\mathrm{d}r_2 d\alpha}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\alpha}}.$$

We can go further and substitute $r_1 \to R_{\mathrm{disk}}r_1'$ (with $\mathrm{d}r_1 = R_{\mathrm{disk}}\mathrm{d}r_1'$) and $r_2 \to R_{\mathrm{disk}}r_2'$ (with $\mathrm{d}r_2 = R_{\mathrm{disk}}\mathrm{d}r_2'$) to obtain

$$U = -\frac{G\mathscr{M}^2}{\pi R_{\mathrm{disk}}}\int_{r_1'=0}^{1}\int_{r_2'=0}^{1}\int_0^{2\pi}\frac{r_1'\mathrm{d}r_1' r_2'\mathrm{d}r_2' d\alpha}{\sqrt{r_1'^2+r_2'^2-2r_1'r_2'\cos\alpha}},$$

where the integral is definite – and independent from the disk properties. Hence, we already know that the result has the expected proportionality:

$$U = -C\frac{G\mathscr{M}^2}{R_{\mathrm{disk}}} \propto R_{\mathrm{disk}}^{-1}.$$

Ballabh (1973, AP&SS, Vol. 24, p.535) computed the gravitational potential energy using Legendre's complete elliptic integral of second kind

$$\mathscr{E} = \int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

with $k = \frac{r}{R_{\mathrm{disk}}}$ as the modulus. His result is

$$U_{\mathrm{disk}} = -2GR_{\mathrm{disk}}\Sigma^2\int_0^{R_{\mathrm{disk}}}\int_0^{2\pi}\mathscr{E}\,r\,dr\,d\varphi = -\frac{8}{3\pi}\frac{G\mathscr{M}^2}{R_{\mathrm{disk}}}$$

Thus $C = 8/(3\pi)$. This value is approximately a factor $\sqrt{2}$ larger than for the gravitational potential of a homogeneous sphere with equal mass and radius.

Problem 2.4 *(2 points)*

From the virial theorem, the condition for the Jeans radius or mass is $K = |U|/2$, where

$$K = \frac{1}{2}\mathscr{M}v^2 = \frac{1}{2}\frac{3kT}{\mu m_{\mathrm{p}}}\mathscr{M} \qquad \text{and} \qquad |U| = \frac{3}{5}\frac{G\mathscr{M}^2}{R}, \tag{8}$$

so that

$$\frac{kT}{\mu m_{\mathrm{p}}}\mathscr{M} = \frac{1}{5}\frac{G\mathscr{M}^2}{R}, \tag{9}$$

giving the Jeans radius

$$R_J = \frac{1}{5}\mu m_{\mathrm{p}}\frac{G\mathscr{M}}{kT}, \tag{10}$$

which contains an additional factor 1/5 (and μ, of course).

We now re-derive the Jeans mass:

$$\mathscr{M} \sim \mu m_{\mathrm{p}}\cdot n\cdot\frac{4}{3}\pi R^3 \qquad \text{or} \qquad R \sim \left(\frac{3}{4}\pi\right)^{1/3}\left(\frac{\mathscr{M}}{n\mu m_{\mathrm{p}}}\right)^{1/3} \tag{11}$$

Substituting this into (9) yields

$$\frac{kT}{\mu m_{\mathrm{p}}} < \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\mathscr{M}^{2/3}(n\mu m_{\mathrm{p}})^{1/3} \tag{12}$$

or

$$kT < \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\mathscr{M}^{2/3} n^{1/3} (\mu m_\mathrm{p})^{4/3} \tag{13}$$

or

$$\mathscr{M}_\mathrm{J} = 5^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}\left(\frac{kT}{Gn^{1/3}(\mu m_\mathrm{p})^{4/3}}\right)^{3/2} \tag{14}$$

or

$$\mathscr{M}_\mathrm{J} = \underbrace{5^{3/2}\left(\frac{3}{4\pi}\right)^{1/2}}_{5.46}\left(\frac{k}{G}\right)^{3/2}\frac{1}{(\mu m_\mathrm{p})^2}\frac{T^{3/2}}{n^{1/2}}. \tag{15}$$

The difference to the result derived in the lecture is the additional factor $\approx 5.5/\mu^2$, which is actually close to unity for a mix of molecular hydrogen (H_2) with a bit of helium (He) with an effective $\mu \approx 2\ldots 2.5$.

Problem 2.5 *(2 points)*

The potential energy of a homogeneous sphere of uniform density with radius R and mass $\mathscr{M}$ is

$$|U| = \frac{3}{5}\frac{G\mathscr{M}^2}{R}$$

The *total* energy of a spherical cloud if an ideal gas is $E = K + U$ or, explicitly,

$$E = \frac{3}{2}kT\frac{\mathscr{M}}{\mu m_\mathrm{p}} - \frac{3}{5}\frac{G\mathscr{M}^2}{R}.$$

As in the initial state $T \approx 0$, $R \approx \infty$, and hence $E \approx 0$, and the energy is conserved ($E = \mathrm{const}$), for the final state we find

$$\frac{3}{2}kT\frac{\mathscr{M}}{\mu m_\mathrm{p}} = \frac{3}{5}\frac{G\mathscr{M}^2}{R}.$$

and

$$T = \frac{2G\mathscr{M}\mu m_\mathrm{p}}{5kR}$$

With $\mu = 2$, this gives (in cgs units)

$$T \approx \frac{4\cdot 7\cdot 10^{-8}\cdot 2\cdot 10^{33}\cdot 2\cdot 10^{-24}}{5\cdot 1.4\cdot 10^{-16}\cdot 200\cdot 1.5\cdot 10^{13}}\ \mathrm{K} \approx 500\ \mathrm{K}.$$

(You obtain similar results when invoking the virial theorem or the Jeans criterion instead, as kinetic and gravitational energy are equated in all three cases, just with slightly different prefactors. However, note that the virial theorem would only apply if we allowed for loss of energy, e. g., due to radiation.)